Dense m-convex Fréchet Subalgebras of Operator Algebra Crossed Products by Lie Groups

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Abstract

Let A be a dense Fréchet *-subalgebra of a C*-algebra B. (We do not require Fréchet algebras to be m-convex.) Let G be a Lie group, not necessarily connected, which acts on both A and B by *-automorphisms, and let σ be a submultiplicative function from G to the nonnegative real numbers. If σ and the action of G on A satisfy certain simple properties, we define a dense Fréchet *-subalgebra $G \rtimes^\sigma A$ of the crossed product $L^1(G, B)$. Our algebra consists of differentiable A-valued functions on G, rapidly vanishing in σ.

We give conditions on the action of G on A which imply the m-convexity of the dense subalgebra $G \rtimes^\sigma A$. A locally convex algebra is said to be m-convex if there is a family of submultiplicative seminorms for the topology of the algebra. The property of m-convexity is important for a Fréchet algebra, and is useful in modern operator theory.

If G acts as a transformation group on a manifold M, we develop a class of dense subalgebras for the crossed product $L^1(G, C_0(M))$, where $C_0(M)$ denotes the continuous functions on M vanishing at infinity with the sup norm topology. We define Schwartz functions $\mathcal{S}(M)$ on M, which are differentiable with respect to some group action on M, and are rapidly vanishing with respect to some scale on M. We then form a dense m-convex Fréchet *-subalgebra $G \rtimes^\sigma \mathcal{S}(M)$ of rapidly vanishing, G-differentiable functions from G to $\mathcal{S}(M)$.

If the reciprocal of σ is in $L^p(G)$ for some p, we prove that our group algebras $\mathcal{S}^\sigma(G)$ are nuclear Fréchet spaces, and that $G \rtimes^\sigma A$ is the projective completion $\mathcal{S}^\sigma(G) \widehat{\otimes} A$.

AMS subject classification: Primary: 46E25, Secondary: 46H35, 46A11, 46K99.

Contents

§0 Introduction

Let G be a Lie group, not necessarily connected, and let B be a C*-algebra with strongly continuous action of G. We let A be a dense Fréchet *-subalgebra of B (with Fréchet topology possibly stronger than the topology on B), such that G acts on A and the action of G on A is infinitely differentiable. (Fréchet algebras are assumed locally convex, but not m-convex.) We construct and study a class of dense Fréchet *-subalgebras $G \rtimes A$ of the convolution algebra $L^1(G, B)$. Many such (topological) subalgebras have been defined in the literature

for various sorts of groups and algebras. The dense subalgebras in [ENN], [Jo], [Ji], [Vi §7], the Schwartz algebras in [Bo, Thm 2.3.3], and the Schwartz algebra of a connected simply connected nilpotent Lie group [Ho], [Lu 1], [Lu 2] are all special cases of the algebras we define here. Our algebras consist of differentiable rapidly vanishing functions from G to A. They have some advantages over algebras of compactly supported differentiable functions. For example, the algebras of compactly supported functions are rarely spectral invariant in the C*-crossed product $G \rtimes B$, unless G is compact or G acts properly on a manifold (see [BC, Appendix 1]). However, the Schwartz algebra of a connected simply connected nilpotent Lie group is always spectral invariant in the group C*-algebra $C^*(G)$ [Lu 1, Prop 2.2] [Sc 3, Cor 7.17]. Spectral invariance is an important property since it implies that the K-theories $K_*(G \rtimes A)$ and $K_*(G \rtimes B)$ are the same [Co, VI.3][Sc 2, Lemma 1.2, Cor 2.3]. In the papers [Sc 1][Sc 2][Sc 3][Sc 4] we study the representation theory and spectral invariance of the dense subalgebras we define here.

We consider the following situation. We call a positive Borel measurable function $\sigma \geq 1$ on G a *weight which bounds Ad* if it satisfies the four properties

$$\begin{aligned}\sigma(gh) &\leq \sigma(g)\sigma(h) \\ \sigma(g^{-1}) &= \sigma(g) \\ \sigma(e) &= 1 \\ \| Ad_g \| &\leq D\sigma^p(g)\end{aligned}$$

for some constant D and some positive integer p, where $g, h \in G$, and $\| \quad \|$ denotes some norm on the space of linear operators on the Lie algebra of G. The group algebra $\mathcal{S}^\sigma(G)$ of σ-rapidly vanishing differentiable functions on G is then well formed, and is a dense Fréchet *-subalgebra of $L^1(G)$. (This corresponds to the case $A = B = \mathbb{C}$.) In §2, we define an appropriate notion of a tempered action of G on the dense subalgebra A, much as in [ENN] for the case $G = \mathbb{R}$, and [DuC 1, §4][DuC 2] for the case of a simply connected, connected nilpotent Lie group. If the action of G on A is tempered, we define a dense Fréchet *-subalgebra $G \rtimes^\sigma A$ of $L^1(G, B)$, which consists of σ-rapidly vanishing A-valued differentiable functions on G.

As we noted, these dense subalgebras $G \rtimes^\sigma A$ generalize those in [ENN], and [Bo, Thm 2.3.3]. (The smooth crossed products by $\mathbb{R}$ in [ENN, §7] and by $\mathbb{Z}$ in [Ns] may not in general be of the form $G \rtimes^\sigma A$, but they are always a projective limit $\varprojlim G \rtimes^{\sigma_n} A$, where σ_n is an increasing sequence of weights on G.) In the purely group algebra case, our L^2 version of

Schwartz functions $\mathcal{S}_2^\sigma(G)$ generalizes the spaces of rapidly vanish functions on a discrete group with a length function studied in [Ji] [Jo] [Vi, §7] (If G is rapidly decaying [Jo], these L^2 Schwartz functions become algebras.) See Remark 6.18 below for a comparison of our algebras with the group algebra of Rader defined for a reductive Lie group [War], and with the zero Schwartz space for $SL_2(\mathbb{R})$ studied in [Bar, §19].

We investigate whether our dense subalgebras are m-convex, that is, whether their topology is given by a family of submultiplicative seminorms. The property of m-convexity for Fréchet (or more general locally convex) algebras is important for several reasons. Arens was the first to notice that m-convexity implies the existence and continuity of entire functions on the algebra [Ar]. Although a unital m-convex algebra may not have an open group of invertible elements, inversion is always continuous on the set of invertible elements [Mi, Prop 2.8]. Every m-convex algebra is a projective limit of Banach algebras [Mi, Thm 5.1]. Many other important properties of m-convexity are studied in [Mi] and [Ze]. The property of m-convexity often arises in modern operator algebra theory [Ph], [Br], [Da, p. 136]. For example, in my thesis [Sc 1] I study a method for showing that dense subalgebras are spectral invariant in their C*-algebras, which requires the dense subalgebras to be m-convex.

We define what it means for an action of G on A to be *m-tempered* (see §3), and show that $G \rtimes^\sigma A$ is m-convex if the action is m-tempered. As far as I know, no results on the m-convexity of a Fréchet crossed product like $G \rtimes^\sigma A$ have appeared in the literature. For the case of the standard Schwartz algebra of a simply connected nilpotent Lie group, the m-convexity is noticed in [Lu 2]. The m-convexity of the group convolution algebra of C^∞-functions with compact support and inductive limit topology is proved in [Ma, Chap VIII, §11]. In §3.2, we also give several interesting examples of *non* m-convex group Schwartz algebras.

In §4, we show that $G \rtimes^\sigma A$ has a well defined and continuous $*$-operation. In §5, we develop the example of a transformation group, where $B = C_0(M)$, the commutative C*-algebra of continuous functions on a locally compact G-space M vanishing at infinity, with pointwise multiplication. We let γ be a scale (or nonnegative real-valued function) on M, and define $C^\gamma(M)$ to be the Fréchet algebra of continuous functions on M which vanish γ-rapidly. Then we take $A = \mathcal{S}_H^\gamma(M)$, the set of C^∞-vectors for the action of a Lie group H on $C^\gamma(M)$. If certain simple conditions for the action of G on M are satisfied (see §5), we show that the convolution algebra $G \rtimes^\sigma \mathcal{S}_H^\gamma(M)$ is a well formed m-convex Fréchet algebra, which is dense in $L^1(G, C_0(M))$.

In §6, we show that the integrability condition

$$(\exists p \in \mathbb{N}) \qquad \int_G \frac{1}{\sigma^p(G)} dg < \infty \tag{0.1}$$

implies that our group algebra $\mathcal{S}^\sigma(G)$ is a nuclear Fréchet space, and that $G \rtimes^\sigma A$ is then isomorphic to the projective completion $\mathcal{S}^\sigma(G)\widehat{\otimes}A$. We show that condition (0.1) is satisfied in several important cases, and compare our algebras with those in [Jo].

We end this introduction by giving a brief overview of some of the dense subalgebras of crossed products we are familiar with from the literature. All of the dense subalgebras we speak of will be complete locally convex algebras in some topology.

First consider the group algebra case, when A and B are both the complex numbers. For a general Lie group, the C^∞-functions with compact support form a dense subalgebra (see [Ma, Chap. VIII §11]). If G is compact, this subalgebra of course reduces to the convolution algebra of C^∞ functions on G. When G is a simply connected nilpotent Lie group, there is the well known algebra of Schwartz functions on G obtained via the exponential map (see for example [Lu 1], [Ho, p. 346]) This generalizes to an algebra of Schwartz functions for any closed subgroup of a nilpotent Lie group. If G is a reductive Lie group, Harish-Chandra developed a space of rapidly vanishing differentiable functions which is an algebra (see Rader's theorem [War 1, Prop. 8.3.7.14]) - see also [Bar]. For rapidly vanishing functions on a discrete group, see the definitions in [Ji], [Jo] and [Vi, §7]. In the appendix of [Jo], a space of rapidly vanishing functions for a general locally compact group with a (subadditive) length function is also studied. This algebra does not take into account any C^∞-structure of the group.

In the special case of a transformation group, namely when the algebra B is continuous functions vanishing at infinity on a manifold M, many definitions of a dense subalgebra have also been made. We could take the convolution algebra of compactly supported continuous functions on $G \times M$ (see [EH, §3]). This ignores any C^∞ structure on G or M, so we could instead take the C^∞ functions with compact support on $G \times M$. If G is discrete, such an algebra is introduced in the appendix of [BC]. Perhaps the most recent contribution in the case of a transformation group is the work of F. du Cloux [DuC 4] for the case where G is an algebraic Lie group and M an algebraic variety.

Next we consider the general case of dense subalgebras for a crossed product, where B can be any C*-algebra and A is any dense Fréchet subalgebra of B. The C^∞-functions with compact support from G to A can be used as a dense subalgebra. In [ENN] and [Ns] an appropriate subalgebra of A-valued Schwartz functions is defined for crossed products of $\mathbb{R}$

and $\mathbb{Z}$ respectively with an arbitrary dense Fréchet subalgebra A on which G acts smoothly in an appropriate sense. If G is elementary Abelian, a convolution algebra of Schwartz functions is studied in [Bo, see §2.1.4], in the case that A is the set of C^∞ vectors for the action of G on B. Also in this work of Bost, a dense subalgebra of exponentially vanishing (but not differentiable) functions is studied for an arbitrary locally compact group G possessing a subadditive length function - see [Bo, §2.3].

Throughout this paper, the notations $\mathbb{N}$, $\mathbb{N}^+$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{T}$ shall be used for the natural numbers with zero, natural numbers without zero, integers, rationals, reals, and the circle group respectively. All of our algebras will be over $\mathbb{C}$. The term norm may be used interchangeably with the term seminorm. If the positive definiteness of a norm is important, we shall state it explicitly. We shall use the notation $C_0(M)$ ($C_0^\infty(M)$) to mean continuous (differentiable) functions vanishing (along with all their derivatives) at ∞ on a locally compact space (Lie group) M with sup norm (and sup norm of derivatives). Compactly supported continuous (differentiable) functions on a locally compact space (differentiable manifold) M shall be denote by $C_c(M)$ ($C_c^\infty(M)$). The term differentiable will always mean infinitely differentiable. We shall often refer to the standard set of Schwartz functions $\mathcal{S}(\mathbb{R})$ on $\mathbb{R}$. This means the set of differentiable functions on $\mathbb{R}$ that vanish at infinity (along with their derivatives) faster than the reciprocal of any polynomial.

All groups will be assumed Hausdorff, and, unless otherwise stated, locally compact. We work with Fréchet spaces. However, some variant of what is said ought to work if the dense subalgebra A were a general complete locally convex algebra.

This paper is a portion of my Ph.D. thesis, written at the University of California at Berkeley, under the supervision of Marc A. Rieffel. I would like to thank Bill Arveson, Joeseph Wolf, and especially Marc Rieffel for many helpful comments and suggestions.

§1 Weights, Gauges and Group Algebras

In this section, we define what it means for a positive real valued function σ on a Lie group G to be a weight or a gauge and state an appropriate additional condition on σ, namely that σ bound Ad, so that the Fréchet space $\mathcal{S}^\sigma(G)$ of differentiable rapidly vanishing functions on G will be a *-subalgebra of the convolution algebra $L^1(G)$. We give general conditions on G which imply the existence of weights or gauges that bound Ad.

§1.1 Scales, Weights, Gauges, Orderings and Equivalence Relations

We define scales, weights, and gauges, and introduce orderings and equivalence relations on them. We introduce the notion of a compactly generated group, and show that such groups are characterized by having a largest gauge (which we call the word gauge) and a largest weight defined on them. The word gauge will be the analog of the length function on a discrete finitely generated group.

Definition 1.1.1. Let M be any topological space. Let σ be a real valued Borel measurable function which maps M to the half open interval $[0, \infty)$. We call σ a *scale* on the space M. We define some special types of scales. Let G be any (Hausdorff) topological group, and let e denote the identity element of G. We say that a scale τ on G is a *gauge* if

$$\tau(gh) \leq \tau(g) + \tau(h) \tag{1.1.2}$$

$$\tau(g^{-1}) = \tau(g) \tag{1.1.3}$$

$$\tau(e) = 0 \tag{1.1.4}$$

for $g, h \in G$. An example of a gauge on the real numbers is given by $\tau(r) = |r|$. The trivial scale $\tau \equiv 0$ defines a gauge on any group. In [Jo] and [Ji], gauges are also called *length functions*, but this differs from the common usage of the term length function in the works [Pr] [Ch] [Ha] [Ly]. In [Pr], what we call a gauge is called a (real valued) semigauge.

We say that a scale ω is a weight [Py] [Dz 1] on G if $\omega \geq 1$ and

$$\omega(gh) \leq \omega(g)\omega(h) \tag{1.1.5}$$

$$\omega(g^{-1}) = \omega(g) \tag{1.1.6}$$

$$\omega(e) = 1 \tag{1.1.7}$$

for $g, h \in G$. By [Dz 1, Prop 2.1] or Theorem 1.2.11 below, if G is locally compact, then weights are bounded on compact sets. Since any gauge τ defines a weight on G by $\omega(g) = e^{\tau(g)}$, gauges are also bounded on compact sets. Two examples of weights on $\mathbb{R}$ are given by $\omega(r) = e^{|r|}$ and $\omega(r) = 1 + |r|$. The trivial scale $\omega \equiv 1$ defines a weight on any group. The correspondences $\tau \mapsto e^{\tau}$ and $\omega \mapsto \log(\omega)$ give a bijection between gauges and weights on a group G. We call this bijection the *exponential bijection* between gauges and weights.

Definition 1.1.8. We say that a scale σ_2 *dominates* another scale σ_1 ($\sigma_2 \succcurlyeq \sigma_1$) if there are constants $C, D \geq 0$ and $m \in \mathbb{N}$ such that

$$\sigma_1(g) \leq C\sigma_2^m(g) + D, \qquad g \in G. \tag{1.1.9}$$

We say that σ_1 and σ_2 are *equivalent* ($\sigma_1 \sim \sigma_2$) if $\sigma_1 \preccurlyeq \sigma_2$ and $\sigma_2 \preccurlyeq \sigma_1$. Since $\preccurlyeq$ is transitive, we have a partial order on equivalence classes of scales. We say that σ_2 *strictly dominates* σ_1 if $\sigma_1 \preccurlyeq \sigma_2$ but σ_1 and σ_2 are not equivalent.

On a compact group, since weights and gauges are bounded, every weight and every gauge is equivalent to the constant scale $\sigma \equiv 1$.

The weight $\omega(r) = 1 + |r|$ is equivalent to the gauge $\tau(r) = |r|$ on $\mathbb{R}$. In fact, every gauge τ on an arbitrary topological group is equivalent to the (subadditive) weight $1 + \tau$. Conversely, if ω is a subadditive weight, we can define an equivalent gauge τ by

$$\tau(g) = \begin{cases} 0 & g = e \\ \omega(g) & g \neq e. \end{cases}$$

Example 1.1.10. In general, if ω is a weight which is not subadditive, there is no way of defining a gauge which is equivalent to ω. For example, let $G = \mathbb{Z}$ and define $\omega(n) = e^{|n|}$. Then if τ is a gauge on $\mathbb{Z}$, the inequality (1.1.9) cannot be satisfied for $\sigma_1 = \omega$ and $\sigma_2 = \tau$, since for positive n, $\omega(n) = e^{|n|}$ and $\tau(n) \leq n\tau(1)$. So ω cannot be equivalent to τ. We thus have the proper inclusion

$$\{\text{gauges}\} = \{\text{subadditive weights}\} \subsetneq \{\text{weights}\}$$

of equivalence classes. In fact, since ω dominates any linear function on $\mathbb{Z}$, it strictly dominates every gauge on $\mathbb{Z}$.

Example 1.1.11. On groups with more than two generators, it is easy to come up with weights which are not equivalent to any gauge, but also do not dominate every gauge. For example, let $G = \mathbb{Z}^2$, and define $\omega(n_1, n_2) = e^{|n_1|}(1 + \log(1 + |n_2|))$. Then by Example 1.1.10, ω restricted to the first copy of $\mathbb{Z}$ strictly dominates every gauge on $\mathbb{Z}$, so ω (as a weight on G) cannot be equivalent to any gauge. However, the weight $1 + \log(1 + |n_2|)$ on the second copy of $\mathbb{Z}$ is strictly dominated by the gauge $|n_2|$, so ω cannot dominate every gauge on G. This example shows that the partial ordering on equivalence classes of weights may not be total, since we may have $\omega_1 \not\preccurlyeq \omega_2$ and $\omega_2 \not\preccurlyeq \omega_1$

Definition 1.1.12. We say that a gauge τ_2 *strongly dominates* a gauge τ_1 ($\tau_2 \succcurlyeq_s \tau_1$) if there are constants $C, D \geq 0$ such that

$$\tau_1(g) \leq C\tau_2(g) + D, \qquad g \in G. \tag{1.1.13}$$

We call $\succcurlyeq$ the *usual ordering* on gauges, in order to distinguish it from the strong ordering $\succcurlyeq_s$. We say that τ_1 and τ_2 are *strongly equivalent* ($\tau_1 \sim_s \tau_2$) if $\tau_1 \preccurlyeq_s \tau_2$ and $\tau_2 \preccurlyeq_s \tau_1$.

The natural exponential bijection between gauges and weights given by $\tau \mapsto e^\tau$ and $\omega \mapsto \log \omega$ is order preserving, when the strong ordering is placed on gauges and the usual ordering on weights. On any compact group, every gauge is bounded by a constant and hence is strongly equivalent to the trivial gauge $\tau \equiv 1$.

Example 1.1.14. We give an example of two gauges which are equivalent but not strongly equivalent. Let $G = \mathbb{Z}$ and let $\tau_1(n) = |n|$, $\tau_2(n) = |n|^{1/2}$. We have $\tau_2 \leq \tau_1$ and $\tau_1 \leq \tau_2^2$, so these two gauges are equivalent. However τ_2 cannot strongly dominate its square τ_1, so they are not strongly equivalent.

Example 1.1.15. We give an example showing that for some groups G there is no largest weight, and no largest gauge (in either ordering). Let $G = F_\infty$ be the free group on countably infinitely many generators. Because the exponential bijection between gauges and weights is order preserving, it suffices to show that for any gauge τ there is another gauge which strongly dominates τ, and which is not equivalent to τ (in the usual ordering). Let c_n be the value of τ on the nth generator g_n, where $n \in \mathbb{N}^+$. Define another gauge γ by letting $\gamma(g_{n_1}^{\alpha_1} \dots g_{n_k}^{\alpha_k})$ be the sum $\sum_{i=1}^k |\alpha_i| e^{c_{n_i}+i}$. Since $\tau(g_{n_1}^{\alpha_1} \dots g_{n_k}^{\alpha_k}) \leq \sum_{i=1}^k |\alpha_i| c_{n_i}$, we have $\tau \leq \gamma$ and so $\tau \preccurlyeq_s \gamma$. On the other hand, we cannot have $\gamma \preccurlyeq_s \tau$ or even $\gamma \preccurlyeq \tau$, since then we would have $d \in \mathbb{N}$ and $C, D > 0$ such that

$$\gamma(g_n) = e^{c_n+n} \leq C c_n^d + D, \qquad n \in \mathbb{N}^+. \tag{1.1.16}$$

But e^{c_n} by itself could only be bounded by $Cc_n^d + D$ if c_n is a bounded sequence. And if c_n is bounded, then the right hand side of (1.1.16) is bounded, so the inequality cannot hold. So γ stricly dominates τ. Hence there is no largest gauge on G, in either of the orderings on gauges, and consequently no largest weight on G.

Example 1.1.17. Word Gauge. Let G be a topological group. Let U be any subset containing the identity of G (not necessarily relatively compact), which satisfies $U = U^{-1}$ (we can always replace U with $U \cup U^{-1}$ to achieve this) and such that

$$\bigcup_{n=1}^{\infty} U^n = G. \tag{1.1.18}$$

We call such a set U a *generating set* for G. We define a gauge τ_U on G by

$$\tau_U(g) = \min\{n \mid g \in U^n\}, \tag{1.1.19}$$

where $U^0 = \{e\}$ (see [Py]). If the generating set U is understood, we simply write τ. Note that the set U could be the entire group, in which case $\tau \equiv 0$.

If G is a discrete group and U is a set of generators together with their inverses, then τ is simply the word length. (This follows straight from the definition (1.1.19).) So in general, τ can be regarded as a kind of generalized word length. We call τ the *word gauge with respect to the generating set* U. We may simply refer to τ as the word gauge, if U is understood.

We call the weight $\omega = e^\tau$ the *exponentiated word weight* with respect to U.

Definition 1.1.20. Compactly Generated Groups. We say that a topological group G is *compactly generated* if there is some relatively compact (rel. comp.) generating set U for G[HR, Def 5.12]. If G is locally compact and compactly generated, then G has an open rel. comp. generating set.

Let τ be the word gauge with respect to an open rel. comp. generating set U. The set of g with $\tau(g) \leq n$ is the open set U^n, so the function τ is measurable (in fact upper-semicontinuous) on G. In general, τ is not continuous, even if U is open. For example take $G = \mathbb{R}$, $U = (-1, 1)$, or let G be any connected non-compact group.

Examples of compactly generated groups are given by discrete finitely generated groups and connected locally compact groups. The free group on infinitely many generators is not compactly generated in the discrete topology. (Note that therefore the class of compactly generated groups is not closed under subgroups.) In Example 1.1.15, we showed that there exists no largest equivalence class of weight or gauge on the free (non-compactly generated) group F_∞. More generally, we have the following theorem.

Theorem 1.1.21. *Let G be a locally compact group. If G is compactly generated, the strong equivalence class of the word gauge does not depend on the choice of rel. comp. generating set U. The group G is compactly generated iff G is a countable union of compact sets and there exists a gauge (namely the word gauge) on G which strongly dominates every other gauge on G. A compactly generated group G is compact iff every gauge on G is bounded. Similar statements are true for weights.*

Also, a quotient of a compactly generated group by a closed normal subgroup is compactly generated. If the connected component G_0 of the identity of G is open, then G is compactly generated if and only if the discrete group G/G_0 is finitely generated.

We remark that if G_0 is not open in the last paragraph of the theorem, the forward implication is false. For example, let G be the product of countably many copies of the compact group $\mathbb{Z}/2\mathbb{Z}$. Then G is compact by Tychonoff's theorem, $G_0 = (0, 0, \ldots)$ is not open, and $G = G/G_0$ is not finitely generated.

Proof of Theorem 1.1.21. Assume that G is compactly generated. Given any open rel. comp. generating sets U, V (such sets exist since G is locally compact), there exist positive integers k and l such that $U \subseteq V^k$ and $V \subseteq U^l$. If $g \in U^n$, then $g \in V^{kn}$. It follows that $\tau_V(g) \leq k\tau_U(g)$. Thus the equivalence class of the gauge does not depend on the choice of generating set.

We show that τ strongly dominates every gauge on G. Let γ be another gauge. Let $C = \sup_{g \in U} \gamma(g)$. (Here we have used G locally compact and Theorem 1.2.11 below or [Dz 1, Prop 2.1] to see that $C < \infty$.) Then if $\tau_U(g) = n$, let $g = g_1 \ldots g_n$ with $g_i \in U$. We have

$$\gamma(g) \leq \gamma(g_1) + \ldots \gamma(g_n) \leq nC = C\tau(g).$$

Hence τ strongly dominates γ.

It follows immediately that the exponentiated word weight ω dominates every weight on G. Also, since $G = \cup_{k=0}^{\infty} U^k$, G is a countable union of compact subsets.

If G is compact, clearly every gauge and every weight on G is bounded, since the word gauge is bounded. Conversely, assume that G is compactly generated and that either every gauge on G is bounded, or that every weight on G is bounded. Then the word gauge τ is a bounded function on G, so $G = U^N$ for some $N \in \mathbb{N}$. Hence G is compact, since U is relatively compact. (The converse did not use G locally compact.)

We show that if G is not compactly generated and is a countable union of compact sets, then G has no gauge that dominates every other gauge. Let τ be a gauge on G. To facilitate our proof, we use the following lemma.

Lemma 1.1.22. *Let τ be a gauge on a topological group G. Then τ is strongly equivalent to a gauge τ' on G such that τ' is integer valued and $\tau'(g) = 0$ implies $g = e$.*

proof. Define $\tau_{int}(g)$ to be the least integer greater than or equal to $\tau(g)$. Then $\tau \leq \tau_{int}$ and $\tau_{int} \leq \tau + 1$ so $\tau \sim_s \tau_{int}$. Define

$$\tau'(g) = \begin{cases} 1 & \tau(g) = 0,\, g \neq e \\ \tau_{int}(g) & \text{otherwise} \end{cases}$$

Then τ' is a gauge since $\tau'(gh) \leq \tau'(g) + \tau'(h)$ still holds if $\tau(gh) = 0$ and $gh \neq e$ (since then $\tau'(g)$ or $\tau'(h)$ is greater than or equal to one). Clearly $\tau \sim_s \tau'$. $\square$

So replace τ with the gauge τ' in the lemma. We construct a gauge γ on G such that $\tau \leq \gamma$ and $\gamma \not\leq \tau$. Define

$$U_m = \{\, g \mid \quad \tau(g) \leq m \quad \}, \tag{1.1.23}$$

and let V_m be increasing compact sets containing e such that $\cup_{m=0}^{\infty} V_m = G$. Define increasing rel. comp. sets $W_m = U_m \cap V_m$. Then $\cup_{m=0}^{\infty} W_m = G$, and since G is not compactly generated, we have $\cup_{k=0}^{\infty} W_m^k \neq G$ for each m. Inductively choose sequences $m_p \in \mathbb{N}^+$ and $g_p \in G$ such that $m_0 = 1$, $g_0 = e$ and

$$g_p \in W_{m_p} - \left(\cup_{k=0}^{\infty} W_{m_{p-1}}^k \right) \tag{1.1.24}$$

for $p \geq 1$. Note that $1 = m_0 < m_1 < \ldots$. Define γ by

$$\gamma(g) = \inf_{h_1 \ldots h_k = g,\, h_i \in W_{m_{p_i}} - W_{m_{p_i - 1}}} \left(\sum_{i=1}^{k} e^{m_{p_i}} \tau(h_i) \right), \tag{1.1.25}$$

where the p_i's are the least integers such that $h_i \in W_{m_{p_i}}$. Since the m_p's tend to infinity as p goes to infinity, every $g \in G$ lies in some W_{m_p}. Hence $\gamma(g) \leq e^{m_p} \tau(g) \leq e^{m_p} m_p$, so γ is a well defined function from G to $[0, \infty)$.

We show that γ is gauge. Let $g, h \in G$. Let $h_1 \ldots h_k = g$ and $\tilde{h}_1 \ldots \tilde{h}_{\tilde{k}} = h$ be such that

$$\sum_{i=1}^{k} e^{m_{p_i}} \tau(h_i) \leq \gamma(g) + \epsilon \tag{1.1.26}$$

and

$$\sum_{i=1}^{\tilde{k}} e^{m_{\tilde{p}_i}} \tau(\tilde{h}_i) \leq \gamma(h) + \epsilon. \tag{1.1.27}$$

Then

$$\gamma(gh) \leq \gamma(g) + \gamma(h) + 2\epsilon \tag{1.1.28}$$

by (1.1.26),(1.1.27) and the definition of $\gamma(gh)$ (1.1.25). Letting ϵ tend to zero, we see that γ is subadditive. The remaining properties of a gauge are not too difficult to show.

We show that $\tau \leq \gamma$. Let $g \in G$. Let $h_1 \ldots h_k = g$ be such that (1.1.26) is satisfied. Then

$$\tau(g) \leq \sum^{k} \tau(h_i) \leq \sum^{k} e^{m_{p_i}} \tau(h_i) \leq \gamma(g) + \epsilon.$$

We let ϵ tend to zero to obtain $\tau \leq \gamma$.

Finally, we show that τ does not dominate γ. Recall $\tau(g_p) \leq m_p$ since $g_p \in W_{m_p}$. It suffices to show that $\gamma(g_p) \geq e^{m_p}$. But if we write $g_p = h_1 \ldots h_k$, then by (1.1.24) at least one h_i is *not* in $W_{m_{p-1}}$. Then $h_i \neq e$ so $\tau(h_i) \geq 1$. Thus

$$\gamma(g_p) \geq e^{m_p} 1.$$

This proves the first paragraph of Theorem 1.21.

Let G be compactly generated with generating set U, and let H be any closed normal subgroup. Let $\pi : G \to G/H$ be the canonical map. Then $\pi(U)$ is a relatively compact neighborhood of the identity in G/H and $\pi(U^n) = \pi(U)^n$, so $\pi(U)$ generates G/H. Note that this implies that G/G_0 is a compactly generated group, which has the discrete topology if G_0 is open, and so is finitely generated in that case. Conversely, assume that G/G_0 is finitely generated. Let $g_1 = e, g_2, \ldots g_k \in G$ be such that $\pi(g_i)$ generate G/G_0. Let V be any relatively compact neighborhood of the identity of G_0. (Here we use the local compactness of G.) Let $U = \cup_{i=1}^k g_i V$. Since G_0 is connected, $\cup_{n=0}^\infty U^n \supseteq \cup_{n=0}^\infty V^n = G_0$[HR, Thm 5.7]. Also, $\cup_{n=0}^\infty U^n$ is a subgroup of G which contains an element of every coset of G/G_0. Hence $\cup_{n=0}^\infty U^n = G$ and G is compactly generated. $\square$

If G is compactly generated, we will say that the *word gauge* on G is the word gauge defined above with a relatively compact generating set.

§1.2 Lie Groups and Schwartz functions

If G is a Lie group and σ is a scale on G, we define the σ-rapidly vanishing Schwartz functions $\mathcal{S}^\sigma(G)$ on G. We find an appropriate condition (namely if σ is equivalent to every one of its left translates) on σ so that $\mathcal{S}^\sigma(G)$ is a Fréchet space on which G acts differentiably by left translation.

Assume that G is a real Lie group, not necessarily connected. That is, assume that the connected component G_0 of the identity of G is open and is a real connected Lie group. Our Lie group G is locally compact, so left Haar measure on G exists. Let $\mathfrak{G}$ be the Lie algebra of G, and let q be the dimension of $\mathfrak{G}$. Let $X_1, \ldots X_q$ be a basis for $\mathfrak{G}$, and let G act strongly continuously on a Fréchet space E, with action denoted by β. If $\gamma \in \mathbb{N}^q$, we define the differential operator X^γ by

$$X^\gamma e = \left(\frac{d}{dt_1}\right)^{\gamma_1} \ldots \left(\frac{d}{dt_q}\right)^{\gamma_q} \beta_{exp(t_1 X_1) \ldots exp(t_q X_q)}(e) \restriction_{t_1 = \ldots t_q = 0}, \tag{1.2.1}$$

where e is a C^∞-vector for the action of G on E. If $\| \quad \|_m$ is a family of seminorms topologizing E, then we topologize the set E^∞ of C^∞-vectors for the action of G on E by $\| e \|_{d,\gamma} = \| X^\gamma e \|_d$. In this topology, E^∞ is a Fréchet space, and is a dense G-invariant subspace of E (see the appendix, Theorem A.2).

Definition 1.2.2. Let σ be a scale on G. Let $\mathcal{S}_1^\sigma(G)$ be the set of differentiable functions $\varphi\colon G \to \mathbb{C}$ satisfying

$$\| \varphi \|_{m,\gamma} = \| \sigma^m X^\gamma \varphi \|_1 = \int_G \sigma^m(g) |X^\gamma \varphi(g)| dg < \infty \tag{1.2.3}$$

for each $\gamma \in \mathbb{N}^q$ and $m \in \mathbb{N}$. Here $X^\gamma \psi$ is defined by formula (1.2.1) with $\beta_g(\varphi)(h) = \varphi(g^{-1}h)$. We call $\mathcal{S}_1^\sigma(G)$ the *σ-rapidly vanishing (L^1) Schwartz functions on G*. Since we shall always use the L^1-norm until §6, we shall simply write $\mathcal{S}^\sigma(G)$ for $\mathcal{S}_1^\sigma(G)$. We topologize $\mathcal{S}^\sigma(G)$ by the seminorms (1.2.3). It is easily checked that if $\sigma_1 \sim \sigma_2$, then $\mathcal{S}^{\sigma_1}(G)$ is isomorphic to $\mathcal{S}^{\sigma_2}(G)$.

Question 1.2.4. If $\mathcal{S}^{\sigma_1}(G)$ is isomorphic to $\mathcal{S}^{\sigma_2}(G)$, then is $\sigma_1 \sim \sigma_2$?

We find a condition on σ for which the action of G by left translation on $\mathcal{S}^\sigma(G)$ will be a well defined automorphism. We use the notation $\varphi_g(h) = \varphi(g^{-1}h)$. Then

$$\| \varphi_g \|_{d,\gamma}^{\sigma_g} = \int_G \sigma_g^d(h) \,|\, (X^\gamma \varphi_g)(h) \,|\, dh$$

By the chain rule (see (2.2.3)), we may write

$$(X^\gamma \varphi_g)(h) = \sum_{\beta \le \gamma} p_\beta((Ad_{g^{-1}})_{ij})(X^\beta \varphi)_g(h), \tag{1.2.5}$$

where $(Ad_{g^{-1}})_{ij}$ is the ijth matrix entry of $Ad_{g^{-1}}$, and p_β is some polynomial. Hence

$$\begin{aligned} \| \varphi_g \|_{d,\gamma}^{\sigma_g} &\le C_g \sum_{\beta \le \gamma} \int_G \sigma_g^d(h) \,|(X^\beta \varphi)_g(h)| dh \\ &= C_g \sum_{\beta \le \gamma} \int_G \sigma^d(h) \,|(X^\beta \varphi)(h)| dh \\ &= C_g \sum_{\beta \le \gamma} \| \varphi \|_{d,\beta}^{\sigma} \, . \end{aligned} \tag{1.2.6}$$

So for fixed g, the map $\varphi \mapsto \varphi_g$ is a continuous linear map $\mathcal{S}^\sigma(G) \to \mathcal{S}^{\sigma_g}(G)$. If σ_g is equivalent to σ, then $\mathcal{S}^\sigma(G) \cong \mathcal{S}^{\sigma_g}(G)$, so $\varphi \mapsto \varphi_g$ will be a continuous isomorphism of the locally convex space $\mathcal{S}^\sigma(G)$, with inverse $\varphi \mapsto \varphi_{g^{-1}}$.

Definition 1.2.7. We say that a scale σ is *translationally equivalent* if $\sigma_g \sim \sigma$ for every $g \in G$. Note that any weight or gauge is translationally equivalent, and that if $\sigma_1 \sim \sigma_2$, then σ_1 is translationally equivalent iff σ_2 is.

Remark 1.2.8. The set of translationally equivalent scales on a group G (which we will denote by σ here) is closed under pointwise addition and multiplication. The zero gauge acts as the identity for addition, and the weight $\omega \equiv 1$ is the identity for multiplication. The set of gauges is an additive submonoid of σ, and the set of weights is a multiplicative submonoid of σ. If we denote by $\sigma/\!\sim$ the set σ moded out by the equivalence relation on scales, then addition and multiplication are still defined on $\sigma/\!\sim$. One could do the same thing with the set of gauges (for addition) and weights (for multiplication).

From now on, assume that σ is translationally equivalent. Let α_g denote the automorphism $\varphi \mapsto \varphi_g$ of $\mathcal{S}^\sigma(G)$. We seek a condition on σ so that the action α of G on $\mathcal{S}^\sigma(G)$ is strongly continuous and infinitely differentiable. By writing elements of $\mathcal{S}^\sigma(G)$ as the sum of their real and imaginary parts, it suffices to consider real valued functions. For small t, we have

(1.2.9)
$$\begin{aligned} \| \varphi_{exptX} - \varphi \|_{d,\gamma} &= \int_G \sigma^d(h) |X^\gamma(\varphi_{exptX} - \varphi)(h)| dh \\ &= \int_G \sigma^d(h) |tX^{\tilde{\gamma}}(\varphi_{expt_h X})(h)| dh \leq |t| \int_G \sigma^d((expt_h X)h) |X^{\tilde{\gamma}} \varphi(h)| dh, \end{aligned}$$

where t_h is a number between $-t$ and t from the mean value theorem, and $\tilde{\gamma}$ is γ with a derivative in the X direction added on. If $\sigma((expt_h X)h)$ is bounded by a polynomial in $\sigma(h)$ *uniformly in* t_h, then clearly 1.2.9 tends to zero as $t \to 0$. Accordingly, we make the following definition.

We say that σ is *uniformly translationally equivalent* if for every compact subset K of G there exists $C, D > 0$, $d \in \mathbb{N}$ depending on K, such that

$$\sigma_g(h) \leq C\sigma^d(h) + D, \qquad h \in G,\, g \in K. \tag{1.2.10}$$

By (1.2.9), we see that if σ is uniformly translationally equivalent, then the action α on $\mathcal{S}^\sigma(G)$ is strongly continuous. Similar calculations show that the action is also differentiable.

Theorem 1.2.11. *Let G be a locally compact group. If a scale σ is translationally equivalent, it is uniformly translationally equivalent. (So weights and gauges are uniformly translationally equivalent.) Hence if σ is translationally equivalent and G is a Lie group, by our preceding*

remarks the action α of G on $\mathcal{S}^\sigma(G)$ by left translation is strongly continuous and differentiable. Also, translationally equivalent scales are bounded an compacts sets. (Hence weights and gauges are bounded on compact sets.)

Proof. (Compare [Dz 1, Prop 2.1].) Let σ be a translationally equivalent scale. By replacing σ with $\max(1,\sigma)$, we may assume $\sigma \geq 1$. Then for each $g \in G$ we have $C_g > 0$ and $d_g \in \mathbb{N}$ such that

$$\sigma_g(h) \leq C_g \sigma^{d_g}(h), \qquad h \in G. \tag{1.2.12}$$

For each g, we may take d_g to be the least positive integer that satisfies (1.2.12) for some $C_g > 0$. Then, for this g and d_g, we let C_g be the least positive integer satisfying (1.2.12).

Note

$$\sigma_{g_1 g_2}(h) \leq C_{g_2} \sigma_{g_1}^{d_{g_2}}(h) \leq C_{g_2} C_{g_1}^{d_{g_2}} \sigma^{d_{g_1} d_{g_2}}(h), \qquad h \in G, \tag{1.2.13}$$

so

$$d_{g_1 g_2} \leq d_{g_1} d_{g_2}. \tag{1.2.14}$$

Define

$$U_m = \{\, g \mid \quad d_g \leq m \quad \}$$

$$V_m = \{\, g \mid \quad C_g \leq m \quad \}$$

for $m \in \mathbb{N}^+$. One easily checks from the measurability of σ that U_m, V_m are measurable subsets of G. Also $e \in U_m \cap V_m$ and

$$\cup_{m=0}^{\infty} (U_m \cap V_m) = G. \tag{1.2.15}$$

Since Haar measure is countably additive, the Haar measure $|U_m \cap V_m|$ is greater than zero for some m. (Here we have used our assumption that G is locally compact for the existence of Haar measure.)

For this m, the interior of $(U_m \cap V_m)^2$ contains an open neighborhood V of the identity in G [Dz 2, p. 17 and 18]. Then for $v \in V$, we have $v = v_1 v_2$ with $v_1, v_2 \in U_m \cap V_m$. Hence $d_v \leq d_{v_1} d_{v_2} \leq m^2$ by 1.2.14, and by (1.2.13),

$$\sigma_v(h) \leq C_{v_2} C_{v_1}^{d_{v_2}} \sigma^{m^2}(h) \leq m^{m+1} \sigma^{m^2}(h), \qquad h \in G \tag{1.2.16}$$

(As far as I know, d_v could be stricly less than m^2, so (1.2.13) gives no bound on C_v. However, this does not interfere with the proof, since we have (1.2.16).) Let L be any compact subset of G, and let $g_1, \ldots g_n \in G$ be such that

$$\cup_{i=1}^{n} g_i V \supseteq L.$$

Say $g \in L$. Then $g = g_i v$ for some $v \in V$ so

$$\sigma_g(h) = \sigma_v(g_i^{-1}h) \leq m^{m+1}\sigma^{m^2}(g_i^{-1}h), \qquad h \in G. \tag{1.2.17}$$

By the translational equivalence of σ, the quantity on the right hand side of (1.2.17) is bounded by a constant times a power of $\sigma(h)$. Hence σ is uniformly translationally equivalent.

To see that translationally equivalent scales are bounded on compact sets, let L be any compact subset of G. Let $C > 0$ and $d \in \mathbb{N}$ be such that

$$\sigma_g(h) \leq C\sigma^d(h), \qquad h \in G,\, g^{-1} \in L. \tag{1.2.18}$$

Set $g = e$ in (1.2.18) and we see that σ is bounded on L. $\square$

Definition 1.2.19. Let σ be any scale. Define the *σ- rapidly vanishing L^1 functions $L_1^\sigma(G)$* on G to be the space of Borel measurable functions $f\colon G \to \mathbb{C}$ such that

$$\| f \|_d = \int_G \sigma^d(g)\,|\,f(g)\,|\,dg \tag{1.2.20}$$

is finite for all $d \in \mathbb{N}$. Then $L_1^\sigma(G)$ is complete for the topology given by the seminorms $\| \quad \|_d$ [Schw, §5]. Until §6, we shall often refer to $L_1^\sigma(G)$ as $L^\sigma(G)$. The following characterization of $\mathcal{S}^\sigma(G)$ is useful (see for example Proposition 2.3.1), and it also gives the completeness of the space $\mathcal{S}^\sigma(G)$.

Theorem 1.2.21. *Let σ be a translationally equivalent scale (for example a weight or a gauge) on a Lie group G. Then the action α of G by left translation is an isomorphism of the locally convex spaces $\mathcal{S}^\sigma(G)$ and $L^\sigma(G)$, and this action of G is differentiable on $\mathcal{S}^\sigma(G)$ and strongly continuous on $L^\sigma(G)$. The Fréchet space of C^∞-vectors $L^\sigma(G)^\infty$ is naturally isomorphic to $\mathcal{S}^\sigma(G)$. Hence $\mathcal{S}^\sigma(G)$ is complete and a Fréchet space.*

Proof. We saw that α acts differentiably on $\mathcal{S}^\sigma(G)$. By the estimate (1.2.6) with $\gamma = 0$, we see that each α_g is an automorphism of $L^\sigma(G)$. Since α is strongly continuous on $\mathcal{S}^\sigma(G)$, and the inclusion map $\mathcal{S}^\sigma(G) \hookrightarrow L^\sigma(G)$ is continuous with dense image, it is easily checked that α acts strongly continuously on $L^\sigma(G)$. Since α acts differentiably on $\mathcal{S}^\sigma(G)$, we have $\mathcal{S}^\sigma(G) \subseteq L^\sigma(G)^\infty$. To prove the theorem, it suffices to show that $L^\sigma(G)^\infty \subseteq \mathcal{S}^\sigma(G)$. See Theorem 2.1.5 for the proof of this. $\square$

§1.3 Group Schwartz Algebras

We give appropriate conditions on the scale σ so that $L^\sigma(G)$ and $\mathcal{S}^\sigma(G)$ are Fréchet *-algebras (not necessarily m-convex). For $L^\sigma(G)$ and $\mathcal{S}^\sigma(G)$ to be Fréchet algebras, it suffices that σ be a weight or a gauge, or more generally that σ be sub-polynomial (see below). For $L^\sigma(G)$ to be a *-algebra, we require that σ be equivalent to the inverse scale $\sigma(g^{-1})$. For $\mathcal{S}^\sigma(G)$ to be a *-algebra, we need the additional condition that σ bounds Ad.

By a *Fréchet algebra*, we mean a Fréchet space with an algebra structure for which the operation of multiplication is continuous. Multiplication is separately continuous if and only if it is jointly continuous [Wa]. We do not require that Fréchet algebras have submultiplicative seminorms (ie that they be m-convex) - see §3.2. A *Fréchet *-algebra* is a Fréchet algebra with a continuous involution.

We define an involution on functions on G by $\varphi^*(g) = \Delta(g)\overline{\varphi}(g^{-1})$, where Δ is the modular function on G. If σ is a scale, we define the *inverse scale* σ_- by $\sigma_-(g) = \sigma(g^{-1})$. We say that σ is *sub-polynomial* if there exists $C > 0$ and $d \in \mathbb{N}$ such that

$$\sigma(gh) \le C(1+\sigma(g))^d(1+\sigma(h))^d, \qquad g, h \in G. \tag{1.3.1}$$

If $\sigma \sim \tau$, then σ is sub-polynomial if and only if τ is. Sub-polynomial scales are translationally equivalent.

Theorem 1.3.2. *If σ is a sub-polynomial scale, then $L^\sigma(G)$ is a Fréchet algebra under convolution. If σ is any scale which is equivalent to its own inverse, then involution is a well defined continuous conjugate linear isomorphism of the Fréchet space $L^\sigma(G)$. In particular, $L^\sigma(G)$ is a Fréchet *-algebra if σ is a sub-polynomial scale such that $\sigma_- \sim \sigma$ (for example, if σ is a weight or a gauge). It is in fact a dense *-subalgebra of $L^1(G)$, and hence of the group C^*-algebra $C^*(G)$ and the reduced group C^*-algebra $C^*_r(G)$.*

Proof. Without loss of generality, assume $\sigma \ge 1$. Since σ is sub-polynomial, there exists $C > 0$ and $d \in \mathbb{N}$ such that $\sigma(gh) \le C\sigma^d(g)\sigma^d(h)$. By a change of variables, we have

$$\| \varphi * \xi \|_m \le \int\int \sigma^m(g)|\varphi(h)\xi(h^{-1}g)|dhdg \le C \| \varphi \|_{dm} \| \xi \|_{dm} .$$

For the statement about involution, first replace σ with the equivalent scale $\max(\sigma, \sigma_-)$. Then

$$\| \varphi^* \|_d^\sigma = \int \sigma^d(g)\,|\varphi^*(g)|\,dg = \int \sigma^d(g^{-1})\,|\varphi(g)|\,dg = \| \varphi \|_d^{\sigma_-} .$$

$\square$

Example 1.3.3. If σ is not a translationally equivalent scale, $L^\sigma(G)$ is still a Fréchet space [Schw, §5]. We show that it is not in general an algebra under convolution. Let $G = \mathbb{Z}$. Define $\sigma(n) = e^{|n|^{|n|}}$. We show that $L^\sigma(G)$ is not an algebra. Let $\varphi(n) = e^{-|2n|^{|n|}}$. Then $\varphi \in L^\sigma(G)$ since $\sigma^d(n)\varphi(n) = e^{d|n|^{|n|}-2^{|n|}|n|^{|n|}} < e^{d-2^{|n|}}$ if $|n|$ is bigger than 1. We show that $\varphi * \varphi \notin L^\sigma(G)$.

$$\begin{aligned}\| \varphi * \varphi \|_1 &= \sum_{n,m\in\mathbb{Z}} \sigma(n)\varphi(m)\varphi(n-m) > \sum_{n=2m,m>0} \sigma(n)\varphi(m)\varphi(n-m) \\ &= \sum_{m>0} e^{|2m|^{|2m|}} e^{-|2m|^{|m|}} e^{-|2m|^{|m|}} = \sum_{m>0} e^{|2m|^{|m|}(|2m|^{|m|}-2)} > \sum_{m>0} e^{|2m|^{|m|}-2}.\end{aligned}$$

The last sum diverges, so $L^\sigma(G)$ is not an algebra. Note that since $G = \mathbb{Z}$ is discrete, $\mathcal{S}^\sigma(G) = L^\sigma(G)$ so $\mathcal{S}^\sigma(G)$ is not an algebra either.

Question 1.3.4. If $L^\sigma(G)$ is a Fréchet algebra, does σ have to be translationally equivalent? Does σ have to be sub-polynomial?

Example 1.3.5. If σ is sub-polynomial, we shall see below in Theorem 1.3.13 that $\mathcal{S}^\sigma(G)$ is a Fréchet algebra. However, we give an example showing that, unlike $L^\sigma(G)$, the algebra $\mathcal{S}^\sigma(G)$ will *not* in general be a *-algebra if $\sigma \sim \sigma_-$. Let G be the $ax+b$ group, namely 2×2 matrices over $\mathbb{R}^2$ of the form

$$g = \begin{pmatrix} e^a & b \\ 0 & 1 \end{pmatrix}, \qquad a, b \in \mathbb{R}. \tag{1.3.6}$$

Haar measure on G is given by $e^{-a}dadb$, where $dadb$ is Lebesgue measure on $\mathbb{R}^2$. The modular function Δ is given by $\Delta(g) = e^a$. (This is consistent with the convention $\int \varphi(g^{-1})dg = \int \Delta(g)\varphi(g)dg$, so that $\| \varphi^* \|_1 = \| \varphi \|_1$.) We let σ be the constant weight $\sigma \equiv 1$. We show that $\mathcal{S}^\sigma(G)$ is not closed under the * operation $\varphi^*(g) = \Delta(g)\overline{\varphi}(g^{-1})$.

Define

$$\varphi(g) = \frac{e^a}{(1+a^2)(1+b^2)},$$

with g as in (1.3.6). We may think of the Lie algebra as 2×2 real matrices with the second row zero. Choose as a basis for the Lie algebra the two matrices

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.3.7}$$

Then

$$\varphi(exp(-t_2X_2)exp(-t_1X_1)g) = \frac{e^{a+t_1}}{(1+(a+t_1)^2)(1+(e^{t_1}(b-1)+1+t_2)^2)},$$

so since $X^\gamma\varphi$ is given by differentiating in t_1 and t_2, and then setting $t_1 = t_2 = 0$ (see (1.2.1) above with $\beta_g(\varphi)(h) = \varphi(g^{-1}h)$), we see that φ and all of its derivatives are integrable against the Haar measure. Hence $\varphi \in \mathcal{S}^\sigma(G)$. Since $\Delta(exp(-tX_2)g) = \Delta(g) = e^a$ is constant in t, we have

$$\begin{aligned} |\, X^{(0,1)}\varphi^*(g)\,| &= \left|\ \frac{d}{dt}\Delta(exp(-tX_2)g)\varphi(g^{-1}exp(tX_2))\,|_{t=0}\ \right| \\ &= |\,\Delta(g)(X^{(0,1)}\varphi)(g^{-1})e^{-a}\,| = |\,(X^{(0,1)}\varphi)(g^{-1})\,|, \end{aligned} \tag{1.3.8}$$

where we used

$$g^{-1}exp(tX_2) = \begin{pmatrix} e^{-a} & e^{-a}(t-b) \\ 0 & 1 \end{pmatrix}$$

and the chain rule for differentiation in the second step. By (1.3.8), we have

$$\begin{aligned} &\| X^{(0,1)}\varphi^* \|_1 = \int_G |(X^{(0,1)}\varphi)(g^{-1})|dg \\ &= \int_G |(X^{(0,1)}\varphi)(g)|\Delta(g)dg = \int_{\mathbb{R}^2} \left|\left[\frac{e^a}{1+a^2}\frac{-2b}{1+b^2}\right]\right| dadb = C\int_{\mathbb{R}^2}\frac{e^a}{1+a^2}da, \end{aligned}$$

which does not converge. Hence $\varphi^* \notin \mathcal{S}^\sigma(G)$ and $\mathcal{S}^\sigma(G)$ is not a *-algebra.

In order that $\mathcal{S}^\sigma(G)$ be a *-algebra, we make the following definition for a scale σ. This definition will also be important for making the smooth crossed product $G \rtimes^\sigma A$ defined in §2 a Fréchet algebra (see for example [Sc 3, Thm 6.29]).

Definition 1.3.9. Let σ be any scale on the Lie group G. We say that σ *bounds Ad* if there exists $p \in \mathbb{N}$ and positive constants C, D such that

$$\| Ad_g \| \leq C\sigma^p(g) + D, \qquad g \in G, \tag{1.3.10}$$

where $\| \quad \|$ denotes the operator norm on the space of linear operators on the Lie algebra of G. If G is any Lie group, then $\omega(g) = \max(\| Ad_g \|, \| Ad_{g^{-1}} \|)$ is a weight on G which bounds Ad. If G is compactly generated, then the exponentiated word length dominates every weight on G (see Theorem 1.1.21) and so bounds Ad. Note that if Ad_g always acts trivially (for example, if G is Abelian or discrete), then any scale σ automatically bounds Ad. If σ is a scale on G which bounds Ad, and H is a closed subgroup of the identity component

of G which is normal in G, then $\overline{\sigma}(g) = \inf_{h\in H}\sigma(gh)$ is a scale on G/H which bounds Ad. Moreover, if σ is a sub-polynomial scale, a weight or a gauge, then $\overline{\sigma}$ is also. For example, if σ is submultiplicative, then

$$\begin{aligned}\overline{\sigma}(g_1g_2) = \inf_{h_1,h_2\in H}\sigma(g_1g_2h_1h_2) &= \inf_{h_1,h_2\in H}\sigma(g_1(g_2h_1g_2^{-1})g_2h_2)\\ &= \inf_{h_1,h_2\in H}\sigma(g_1h_1g_2h_2) \le \overline{\sigma}(g_1)\overline{\sigma}(g_2),\end{aligned}$$

so $\overline{\sigma}$ is submultiplicative. See also Examples 1.3.14-16 below.

If σ is a weight or a gauge that bounds Ad (or more generally if $\sigma \sim \sigma_-$), then clearly σ_- bounds Ad also. We shall frequently use this, since one condition for the smooth crossed product $G \rtimes^\sigma A$ in §2 to be an algebra will be that σ_- bounds Ad.

Example 1.3.11. We show that in the in Example 1.3.5 above in which $\mathcal{S}^\sigma(G)$ is not a *-algebra, the weight σ does not bound Ad. An easy calculation shows that for g as in (1.3.6),

$$Ad_{g^{-1}} = \begin{pmatrix} 1 & 0 \\ -b & e^a \end{pmatrix} \tag{1.3.12}$$

where the matrix is in the basis (1.3.7). Clearly $\sigma \equiv 1$ does not bound Ad.

Theorem 1.3.13. *Let G be a Lie group, and let σ be a sub-polynomial scale on G. Then the Fréchet space $\mathcal{S}^\sigma(G)$ is a Fréchet algebra under convolution. If in addition σ is equivalent to σ_- and σ bounds Ad (for example, if σ is a weight or a gauge that bounds Ad), then $\mathcal{S}^\sigma(G)$ is a Fréchet *-algebra. It is in fact a dense *-subalgebra of $L^1(G)$, and hence of the group C^*-algebra $C^*(G)$ and the reduced group C^*-algebra $C^*_r(G)$.*

Proof. See proof of Theorem 2.2.6 and Corollary 4.9 setting $A = \mathbb{C}$. $\square$

We give some examples. First, we note that if we take $G = \mathbb{R}$ and let $\omega(r) = 1 + |r|$, then the Schwartz algebra $\mathcal{S}^\omega(G)$ is precisely the standard convolution algebra of Schwartz functions on $\mathbb{R}$.

Example 1.3.14. Let $G = \mathbb{R}$. Define $\omega(r) = e^{|r|}$. This is a weight on G which bounds Ad. The Schwartz algebra $\mathcal{S}^\omega(G)$ is smaller than the standard Schwartz algebra $\mathcal{S}(\mathbb{R})$. It is the same set of functions as $\mathcal{S}_{exp}(\mathbb{R})$ in [DuC 3, §2.1.5]. (In this case, it is irrelevant whether we use the L^1 norm or the sup norm which du Cloux uses—see Theorem 6.8.)

Example 1.3.15. If H is any subgroup of G, such that the inclusion map $H \hookrightarrow G$ is differentiable, then any scale, weight or gauge on G which bounds Ad restricts to a scale, weight or gauge on H which bounds Ad. This is because for $h \in H$, the linear operator Ad_h on the Lie algebra of H is just the restriction of Ad_h as a linear operator on the Lie algebra of G. This allows us to define various gauges or weights (which bound Ad) on groups which are not compactly generated, since such groups may occur as subgroups of compactly generated groups. For example, the free group on infinitely many generators is a subgroup of the compactly generated group $SL(2, \mathbb{R})$, or of the free group on two generators.

Example 1.3.16. Let G be a compact Lie group. Then every scale σ bounds Ad and is equivalent to the zero scale, and $\mathcal{S}^\sigma(G)$ is the Fréchet space $C^\infty(G)$ of differentiable functions on G. This is always a Fréchet *-algebra under convolution.

§1.4 Gauges that bound Ad

As we have noted, there always exists a weight on G which bounds Ad. In a later paper [Sc 3], it will be important to know when there exists a gauge on G which bounds Ad, since these will yield Schwartz algebras which are spectral invariant [Sc 2] [Sc 3] in $L^1(G)$, and in $C^*(G)$. We have noted many examples of groups which do possess gauges which bound Ad - for example the zero gauge bounds Ad if G is discrete, Abelian, or compact.

We note that gauges which bound Ad do not always exist. Assume there is some $h \in G$ such that Ad_h has an eigenvalue λ not lying on the unit circle. Let Y be the corresponding eigenvector. Then

$$\| Ad_{h^n} Y \| = \| e^{n\lambda} Y \| = e^{nRe\lambda} \| Y \| .$$

Let $\epsilon = Re\lambda$. By replacing h with h^{-1}, we may assume that $\epsilon > 0$. Assume that τ is a gauge on G which bounds Ad, and let $C, D > 0$, $d \in \mathbb{N}$ be such that $\| Ad_g \| \leq C\tau^d(g) + D$. Then

$$C\tau^d(h^n) + D \geq \| Ad_{h^n} \| \geq e^{n\epsilon}, \tag{1.4.1}$$

so $\tau(h^n) \geq C^{1/d}(e^{n\epsilon} - D)^{1/d}$. But $\tau(h^n) \leq n\tau(h)$ by subadditivity. This is a contradiction. Hence G has no gauge (or even a subadditive scale) which bounds Ad. An example of such a group is given by the $ax + b$ group above. By (1.3.12), it is clear that the eigenvalues of Ad_g are not always on the unit circle.

Definition 1.4.2. *We say the a Lie group G is Type R if Ad_g has eigenvalues of complex modulus one for all $g \in G$.*

We shall show below (see Corollary 1.4.9) that this generalizes the notion of a Type R connected Lie group [Je] [Pa], which says that the eigenvalues of *ad* are imaginary. In the connected case, a Lie group is Type R if and only if it has polynomial growth [Je][Pa] (see §1.5 below). Nilpotent Lie groups, the Mautner group, motion groups [Pa, p. 229], and any discrete group are all examples of Type R Lie groups. If H is any subgroup of G, such that the inclusion map $H \hookrightarrow G$ is differentiable, then H is Type R if G is Type R. If G is a Type R group, and H is a closed subgroup of the identity component of G which is normal in G, then G/H is Type R.

Theorem 1.4.3. *Let G be a compactly generated Lie group. Assume that the quotient of G by the kernel of Ad has a cocompact solvable subgroup (in particular, it suffices that G have a cocompact solvable subgroup), or assume that G is connected. Then the following are equivalent.*

(1) *The group G is Type R.*
(2) *The word gauge bounds Ad.*
(3) *There exists a gauge on G which bounds Ad.*

The implication (3) $\Rightarrow$ (1) *is true for any Lie group.*

Proof. We have already seen that (3) $\Rightarrow$ (1) above. Also (2) $\Leftrightarrow$ (3) is trivial, since the word gauge dominates every gauge (see Theorem 1.1.21). For (1) $\Rightarrow$ (2), we first prove two lemmas.

Lemma 1.4.4 (Compare [Lu 1, Lemma 1.5]**).** *For $q \in \mathbb{N}$, let G be the group of upper triangular matrices $T_1(q, \mathbb{C})$ with 1's on the diagonal. Then G has a gauge (the word gauge) τ which is equivalent to the weight given by the norm on matrices.*

Proof. For $g \in G$, let $\| g \|_\infty$ denote the maximum absolute value of the *off diagonal* matrix entries $|g_{ij}|$. Let U be any symmetric open neighborhood of the identity of G, such that $U \subseteq \{ g \mid \ \| g \|_\infty \le 1 \ \}$. Let τ be the word gauge with respect to U. We show that τ bounds the norm $\| \ \|_\infty$.

First by induction on n, we have for $1 < i < k < q$,

$$(g_1 \dots g_n)_{ik} = \sum_{i=i_1 \le \dots i_{n+1}=k} (g_1)_{i_1 i_2} \dots (g_n)_{i_n i_{n+1}}, \tag{1.4.5}$$

where $g_1, \dots, g_n \in G$.

Assume that $\tau(g) = n$ for some $g = g_1 \dots g_n$, with $g_i \in U$. Then we have

$$\begin{aligned} \| g \|_\infty &\leq \max_{1<i<k<q} |(g_1 \dots g_n)_{ik}| \leq \max_{1<i<k<q} \left(\sum_{i=i_1 \leq \dots i_{n+1}=k} (1) \right) \leq \sum_{i=1_1 \leq \dots i_{n+1}=q} (1) \\ &\leq \# \text{ of possible ways to arrange steps times } \# \text{ of possible step sizes} \\ &= \left(\binom{n-1}{q-1} + \dots \binom{n-1}{1} \right) q = P_q(n) = P_q(\tau(g)), \end{aligned}$$

where P_q is a polynomial, of degree q, depending only on q. Thus τ dominates the sup norm $\| \quad \|_\infty$.

We show that τ is dominated by the operator norm $\| \quad \|$ on matrices. Let $C > 1$ be such that $\{ g \quad | \quad \| g \| \leq C \} \subseteq U$. Let $T_0(n, \mathbb{C})$ denote the upper triangular complex matrices with 0's on the diagonal. Recall that

$$exp \colon T_0(q, \mathbb{C}) \to G$$

is a polynomial map with polynomial inverse. In fact,

$$\| exph \| \leq e^{\|h\|}, \qquad h \in T_0(q, \mathbb{C}),$$

and

$$\| log(g) \| \leq P(\| g \|), \qquad g \in G,$$

where P is a polynomial. Let $g \in G$ and $h = \log(g)$. Let d be the least integer greater than or equal to $\| h \| / \log(C)$. Then $\| e^{h/d} \| \leq e^{\|h\|/d} \leq e^{\log(C)} = C$ so $\tau(e^{h/d}) \leq 1$. Thus

$$\tau(g) = \tau((e^{h/d})^d) \leq d \leq \| h \| / \log(C) + 1 \leq P(\| g \|)/\log(C) + 1.$$

This proves Lemma 1.4.4. $\square$

We say that a topological group H is a *subgroup* of a topological group G if $H \subseteq G$ with continuous inclusion. We say that a scale σ is a *one-sided gauge* iff σ satisfies the inequality

$$\sigma(gh) \leq C + D\sigma^d(g) + \sigma(h), \qquad g, h \in G,$$

for some $C, D > 0$ and $d \in \mathbb{N}$. Note that one-sided gauges are translationally equivalent and therefore bounded on compact subsets by Theorem 1.2.11.

Lemma 1.4.6. *Let G be any separable subgroup of $GL(q, \mathbb{C})$ and assume that G has a closed cocompact subgroup H with a one-sided gauge which is equivalent to the matrix norm inherited from $GL(q, \mathbb{C})$. Then G has a one-sided gauge which is equivalent to the matrix norm inherited from $GL(q, \mathbb{C})$.*

Proof. Let $B \subseteq G$ be a rel. comp. measurable cross section for the cosets G/H [Pa, App. C]. (Here we have used H closed (to get G/H Hausdorff) and G separable.) Let τ be a one-sided gauge on H which is equivalent to the norm on matrices. Let $C, D > 0$ and $d \in \mathbb{N}$ be such that

$$\tau(h_1 h_2) \leq C + D\tau^d(h_1) + \tau(h_2), \qquad h_1, h_2 \in H.$$

Let $g \in G$, and write $g = bh$ with $b \in B$, $h \in H$. Define

$$\gamma(g) \equiv \tau(h).$$

Since B is measurable, γ is measurable and hence a scale on G.

We show that γ is a one sided gauge. Since $B^{-1}B$ and B are rel. comp. in G, they are both rel. comp. and hence norm bounded sets in $GL(q, \mathbb{C})$. Thus, since τ is equivalent to the norm on matrices, there exists a polynomial P such that

$$\tau(b_1 h b_2) \leq P(\tau(h))$$

for all $h \in H$ and all $b_1 \in B^{-1}B$ and $b_2 \in B$ for which the product $b_1 h b_2$ is in H. Let $g_1, g_2 \in G$ and write $g_1 = b_1 h_1$, $g_2 = b_2 h_2$, $g_1 g_2 = b_3 h_3$. Then

$$\begin{aligned} \gamma(g_1 g_2) &= \tau(h_3) = \tau(b_3^{-1} b_1 h_1 b_2 h_2) \leq C + D\tau^d(b_3^{-1} b_1 h_1 b_2) + \tau(h_2) \\ &\leq C + D(P(\tau(h_1))^d + \tau(h_2) = C + D(P(\gamma(g_1))^d + \gamma(g_2) \leq C' + D'\gamma^k(g_1) + \gamma(g_2) \end{aligned}$$

for some $k \in \mathbb{N}$ and $C', D' > 0$. So γ is a one-sided gauge on G.

A similar argument using the polynomial P and the fact that τ is equivalent to the norm on matrices, shows that γ is equivalent to the norm on matrices. $\square$

Finally, we prove (1) $\Rightarrow$ (2) of Theorem 1.4.3 using Lemmas 1.4.4 and 1.4.6. Let G be any compactly generated Lie group. Let G' denote the image of G in $GL(q, \mathbb{C})$ via the Ad map, where Ad_g acts on the complexification of the Lie algebra. Then G' has a cocompact (closed since we may take the closure of G'' in G') solvable subgroup G'' by assumption. (We shall do the connected case later.) Then G'' has a subgroup G''' of finite index in G'' which

R, each triangular matrix in G''' has entries of modulus one along the diagonal. By Lemmas 1.4.4 and 1.4.6, there is a one-sided gauge equivalent to the norm on matrices on the group of upper triangular matrices in $GL(q,\mathbb{C})$, with entries of modulus one on the diagonal. By restriction, we obtain a one-sided gauge on G''' which is equivalent to the norm on matrices. Since G''' is cocompact and closed in G', and since G' is separable (because G is a compactly generated Lie group, and so separable), we may apply Lemma 1.4.6 to get a one-sided gauge on G' which is equivalent to the norm on matrices. We pull this gauge back to G to get a one-sided gauge on G which bounds Ad. It remains to show that the word gauge dominates this one-sided gauge. Let γ be the one-sided gauge on G, and let τ be the word gauge with respect to a generating set U. Let $K = \sup\{\gamma(g) \,|\, g \in U\}$. Then $K < \infty$ by Theorem 1.2.11, and we have

$$\begin{aligned}\gamma(g_1 \dots g_n) &\leq (n-1)C + D\gamma^d(g_1) + \dots D\gamma^d(g_{n-1}) + \gamma(g_n) \\ &\leq nC + DK^d(n-1) + K = \tau(g)C + DK^d(\tau(g)-1) + K\end{aligned}$$

if $\tau(g) = n$. Hence τ strongly dominates γ, and τ bounds Ad.

Finally, assume G is connected and Type R. Then by [Pa, Prop 6.29], G has cocompact radical. Hence G has a cocompact solvable subgroup, and so does the image of G via Ad. This proves Theorem 1.4.3 $\square$

Question 1.4.7. Is it true for a general compactly generated Lie group G that G has a gauge which bounds Ad iff G is Type R ?

Corollary 1.4.8. *Assume that G is a compactly generated Lie group. If G has a closed cocompact solvable subgroup H, such that for every $h \in H$ the eigenvalues of Ad_h, acting an operator on the Lie algebra of G, lie on the unit circle, then G is Type R and has a gauge that bounds Ad.*

Proof. Let G_1 be the image of G in $GL(q,\mathbb{C})$ via the Ad representation. Let H_1 denote the image of the cocompact subgroup H via this same representation. Replace H_1 with its closure in G_1 if necessary. Then H_1 is a subgroup of $GL(q,\mathbb{C})$ which is solvable and all of whose eigenvalues lie on the unit circle. By Lemmas 1.4.4 and 1.4.6 above, H_1 has a one-sided gauge which is equivalent to the norm on matrices. Hence by Lemma 1.4.6, G_1 has a one-sided gauge that is equivalent to the norm on matrices. We pull this back to G to get a one-sided gauge which bounds Ad. Then, since G is compactly generated, we saw above (at

the end of the proof of Theorem 1.4.3) that the word gauge dominates this one-sided gauge. Therefore the word gauge bounds Ad. Since the existence of a gauge that bounds Ad implies that G is Type R, we are done. □

We say that a connected Lie group G is ad Type R if all the eigenvalues of ad are imaginary.

Corollary 1.4.9. *A connected Lie group is Type R if and only if it is ad Type R.*

Proof. Clearly Type R implies ad Type R if G is connected, since the spectrum of $Ad_{expX} = e^{adX}$ is just the set $e^{spec(ad(X))}$. Assume that G is ad Type R. Let H denote the radical of G, and let $\mathfrak{H}$ denote the Lie algebra of H. Then the set $\{expX \mid X \in \mathrm{H}\}$ is dense in H by [Pa, App E]. Since G is ad Type R, Ad_{expX}, as an operator on the Lie algebra $\mathfrak{G}$ of G, has all its eigenvalues lying on the unit circle. Since for $h \in H$, Ad_h is the norm limit of linear operators on $\mathfrak{G}$ having eigenvalues on the unit circle, Ad_h must also have eigenvalues on the unit circle. By [Pa, Prop 6.29], H is cocompact in G. Hence by Corollary 1.4.8, G is Type R. □

§1.5 Polynomial Growth Groups

We exhibit a large class of Type R groups, all of which have gauges that bound Ad. We say that a locally compact group has *polynomial growth* if for every relatively compact neighborhood U of the identity, the Haar measure $|U^n|$ is bounded by a polynomial in n[Pa]. Examples of polynomial growth groups are given by finitely generated polynomial growth discrete groups [Ji], closed subgroups of nilpotent Lie groups, connected Type R Lie groups, motion groups, the Mautner group, and compact groups.

Proposition 1.5.1. *Let G be a locally compact group. If G has a gauge τ that satisfies the integrability condition*

$$(1.5.2) \qquad (\exists p \in \mathbb{N}) \quad \int_G \frac{1}{(1+\tau(g))^p} dg < \infty,$$

then G has polynomial growth. If G is compactly generated, then G has polynomial growth if and only if G has a gauge which satisfies (1.5.2). If G has a gauge τ that satisfies (1.5.2) and H is a closed subgroup of G, then the restriction of τ to H also satisfies (1.5.2) for some possibly larger $p \in \mathbb{N}$.

Proof. Let τ be a gauge on G which satisfies (1.5.2). Let U be any rel. comp. neighborhood

of e. Then for $n \in \mathbb{N}$,

$$\int_{U^n} dg \leq \sup_{g \in U^n} (1+\tau(g))^p \int_G \frac{1}{(1+\tau(g))^p} dg \leq C\big(1 + n \sup_{g \in U} \tau(g)\big)^p \leq C(1+nD)^p,$$

where we have used the fact that gauges are bounded on compact sets (see Theorem 1.2.11). So G has polynomial growth.

Assume G is compactly generated and has polynomial growth. Let U be a generating set, and let τ be the word gauge. Let p be such that $|U^n| \leq C(1+n)^p$. Then

$$\begin{aligned}\int_G \frac{1}{(1+\tau(g))^{p+2}} dg &\leq \sum_{n=0}^{\infty} \int_{U^n - U^{n-1}} \frac{1}{(1+\tau(g))^{p+2}} dg \\ &\leq \sum_{n=0}^{\infty} \frac{|U^n|}{(1+n)^{p+2}} \leq \sum_{n=0}^{\infty} \frac{C}{(1+n)^2} < \infty.\end{aligned}$$

For the last statement of the proposition, see Proposition 6.13 below. □

Question 1.5.3. Does a polynomial growth group always have a gauge which satisfies the integrability condition (1.5.2) ?

Lemma 1.5.4[Pa, Prop 6.6,6.9]**.** *If G has polynomial growth, then G is unimodular.*

The following proposition is similar to [Pa, Prop 6.20(ii)].

Proposition 1.5.5. *Let G be a locally compact compactly generated polynomial growth group with closed normal subgroup H. Assume that the identity component G_0 is open in G. Then the quotient group G/H has polynomial growth.*

Proof. By [Bou, chap VII, §2, thm 2],

$$\int_G f(g)dg = \int_{G/H} \int_H f(gh) dh dg, \tag{1.5.6}$$

for $f \in L^1(G)$. If $S \subseteq H$, define

$$\lambda_H(S) = \int_S dh.$$

Let $V = V^{-1}$ be a generating set for G. Let $U = \pi(V)$, where $\pi\colon G \to G/H$ is the canonical map. For $\delta > 0$, define

$$U_\delta = \{\pi(g) \in G/H \,|\, \lambda_H(g^{-1}V \cap H) \geq \delta \quad \text{and} \quad \lambda_H(gV \cap H) \geq \delta,\ g \in G\}.$$

If $\pi(g) \notin U$, then $g^{-1}V \cap H = \varnothing$. So $U_\delta \subseteq U$ for each $\delta > 0$. Also, if $\pi(g) \in U$, then $g^{-1}V$ and gV are open sets whose intersection with H is nontrivial. Since H is a closed subgroup

of G (with relative topology), the sets $g^{-1}V \cap H$ and $gV \cap H$ are nonempty open sets in H. Hence $\lambda_H(g^{-1}V \cap H) > 0$ and $\lambda_H(gV \cap H) > 0$. It follows that

$$\bigcup_{\delta>0} U_\delta = U. \tag{1.5.7}$$

For sufficiently small δ, U_δ has nonzero measure in G/H (by (1.5.7) and since U is a generating set for G/H by Theorem 1.1.21). It follows that U_δ^2 contains a neighborhood of zero [Dz 2, p.17-18]. Since $U_\delta = U_\delta^{-1}$, the set $\bigcup_{n=0}^\infty U_\delta^n$ is a nonempty open subgroup of G/H, and hence must contain the connected component of the identity of G/H. We show that for δ sufficiently small, U_δ also generates all of G/H. Let q be the canonical projection

$$q\colon G/H \to (G/H)/(G/H)_0.$$

By (1.5.7) and since $(G/H)/(G/H)_0$ is discrete, for δ sufficiently small we must have $q(U_\delta) = q(U)$. (To obtain the discreteness, we used that $(G/H)_0$ is open, which follows from our assumption that G_0 is open.) Then $q(U_\delta)$ generates the discrete group $(G/H)/(G/H)_0$. It follows that U_δ generates G/H. From now on, we fix δ sufficiently small so that U_δ generates G/H.

To see that G/H has polynomial growth, it suffices to show that the Haar measure of U_δ^n is bounded by a polynomial in n for $n \in \mathbb{N}$. To do this, we first show that if $\pi(g) \in U_\delta^n$, then $\lambda_H(g^{-1}V^n \cap H) \geq \delta$. Let $g = g_1 \dots g_n$ where $\pi(g_i) \in U_\delta$. Then $g_i = \tilde{g}_i h_i$ where $\tilde{g}_i \in V$ and $h_i \in H$. So, using the fact that H is a normal subgroup, $g^{-1}V^n \cap H = \tilde{h}\tilde{g}_n^{-1} \dots \tilde{g}_1^{-1} V^n \cap H \supseteq \tilde{h}\tilde{g}_n^{-1}V \cap H$ for some $\tilde{h} \in H$. By left invariance of Haar measure on H,

$$\lambda_H(g^{-1}V^n \cap H) \geq \lambda_H(\tilde{g}_n^{-1}V \cap H) \geq \delta, \tag{1.5.8}$$

since $\pi(\tilde{g}_n) = \pi(g_n) \in U_\delta$.

If S is a set, we let χ_S denote the characteristic function of S. We estimate

$$\begin{aligned} \int_{V^n} dg &= \int_G \chi_{V^n}(g)dg = \int_{G/H}\int_H \chi_{V^n}(gh)dhdg \\ &= \int_{G/H} \lambda_H(g^{-1}V^n \cap H)dg \geq \int_{U_\delta^n} \lambda_H(g^{-1}V^n \cap H)dg \geq \int_{U_\delta^n} \delta dg, \end{aligned} \tag{1.5.9}$$

where we used (1.5.8) in the last step. The last expression in (1.5.9) is the Haar measure of U_δ^n in G/H times δ. Since U_δ is a generating set, G/H has polynomial growth. $\square$

Theorem 1.5.10([Lo, Cor to Thm 2])**.** *Let G be a compactly generated polynomial growth Lie group. Then there exists a normal series of closed subgroups $G_1 \lhd G_2 \lhd G$ such that G_1 is a connected solvable polynomial growth Lie group, G_2/G_1 is a nilpotent finitely generated discrete group, and G/G_2 is compact.*

Proof. See [Lo, Cor to Thm 2] and its proof (last paragraph of [Lo]). $\square$

Corollary 1.5.11. *Let G be a compactly generated polynomial growth Lie group. Then any closed subgroup of G is compactly generated. In particular, closed subgroups of compactly generated polynomial growth Lie groups are compactly generated polynomial growth Lie groups.*

Proof. The second statement follows from the first and Proposition 1.5.1. By [BWY, Thm 2.11, Cor 2] it suffices to show that every closed subgroup of the group G_2 from Theorem 1.5.10 is compactly generated.

Note that since G_2/G_1 is a finitely generated nilpotent discrete group, any closed subgroup of it is finitely generated. Also, by [BWY, Prop 2.1, Cor 3] and since G_1 is solvable and connected, every closed subgroup of G_1 is compactly generated. Let F be any closed subgroup of G_2. We apply [BWY, Prop 2.1] to see that if FG_1 is closed in G_2, then F is compactly generated.

It remains to show that FG_1 is closed. If $f_n g_n \to g_0 \in G_2$, then $f_n g_n$ must eventually lie in the same G_1 coset c of G_2. (Recall that G_1 is the connected component of the identity of G_2.) But $c \cap FG_1 = c$, so $g_0 \in FG_1$. $\square$

Corollary 1.5.12. *If G is a compactly generated Type R polynomial growth Lie group, then G has a gauge which bounds Ad.*

Proof. By Theorem 1.5.10, we know G has a cocompact closed solvable subgroup H. Since G is Type R, the conclusion follows from Theorem 1.4.3. $\square$

Every closed subgroup of a connected nilpotent Lie group satisfies the hypotheses of Corollary 1.5.12. So do all connected polynomial growth Lie groups, and hence the Mautner group and all motion groups.

It is not true in general that a Lie group is Type R if and only if it has polynomial growth. For example, any discrete group is Type R. An example of a polynomial growth group that is not Type R is given in [Lo, Ex 1]. However, we have the following theorem.

Theorem 1.5.13. *Let G be a compactly generated polynomial growth Lie group, and assume that the connected component of the identity G_0 is solvable and simply connected. Then G is*

Type R and has a gauge that bounds Ad.

Proof. The maximal compact subgroup of the center of G_0 is trivial, so G is Type R by [Lo, Thm 1 $a) \Rightarrow b)$]. $\square$

Example 1.5.14. If G is nilpotent and connected, then the Schwartz algebra $\mathcal{S}(G)$ we obtain using the word gauge agrees with the usual notion of the Schwartz algebra of G [Lu 1], [Ho, p. 346]. In this case, G is easily seen to be Type R, and has polynomial growth by [Pa, Thm 6.17, 6.39].

Example 1.5.15. Let G be the three dimensional Heisenberg Lie group, that is the set of matrices of the form

$$g = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix},$$

where $a, b, c \in \mathbb{R}$. Then G satisfies the hypotheses of Theorem 1.5.13. The word gauge is equivalent to the scale $\tau(g) = |a| + |c| + |b|^{1/2} + |b - ca|^{1/2}$. We omit the proof of this, since the argument is the same as provided for the $ax + b$ group below in Example 1.6.1(or see [Lu 1, Lemma 1.5]). A quick calculation shows that $\tau(e) = 0$, $\tau(g^{-1}) = \tau(g)$ and $\tau(gh) \leq 3(\tau(g) + \tau(h))$, so τ is very much like a gauge. In an appropriate basis for the Lie algebra, we have

$$Ad_g = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}.$$

One can then see directly that τ bounds Ad.

§1.6 Examples of Weights that Bound Ad

Now we look at several groups which are not Type R. Our primary concern will be to give explicit formulas for the exponentiated word weight, which we will refer to as the word weight for brevity.

Example 1.6.1. Let G be the $ax + b$ group (see Example 1.3.5 above). Define $\omega(g) = e^{|a|} + |e^{-a}b| + |b| + 1$. Then ω is a continuous weight on G, which clearly bounds Ad by (1.3.12). We may form the Schwartz algebra $\mathcal{S}^\omega(G)$. This is the same set of functions as the $\mathcal{S}$ of [DuC 3, §5.2.5]. (Again, du Cloux uses the sup norm, and we use the L^1 norm, but we shall see that this makes no difference by Theorem 6.8 and Proposition 6.13 (1).) We have seen that since G is not Type R, ω is not equivalent to any gauge.

We show that ω is equivalent to the word weight. Let τ be the word gauge. We get an upper bound on τ. Let U be a generating set containing all matrices of the form

$$\begin{pmatrix} e^a & b \\ 0 & 1 \end{pmatrix}, \qquad |a|, \quad |b| \leq 1.$$

By the definition of τ, we have

$$\tau(g) = \tau \begin{pmatrix} e^a & b \\ 0 & 1 \end{pmatrix} \leq \tau \begin{pmatrix} e^a & 0 \\ 0 & 1 \end{pmatrix} + \tau \begin{pmatrix} 1 & e^{-a}b \\ 0 & 1 \end{pmatrix} \leq |a| + \tau \begin{pmatrix} 1 & e^{-a}b \\ 0 & 1 \end{pmatrix}. \tag{1.6.2}$$

Choose any $n \in \mathbb{N}$ for which there exists a γ satisfying $1/e < |\gamma| \leq 1$ and

$$\begin{pmatrix} 1 & (e^{n-1} + \ldots 1)\gamma \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & e^{-a}b \\ 0 & 1 \end{pmatrix}. \tag{1.6.3}$$

Then $\begin{pmatrix} 1 & e^{-a}b \\ 0 & 1 \end{pmatrix} \in U^{2n}$ since

$$\begin{pmatrix} 1 & (e^{n-1} + \ldots 1)\gamma \\ 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} e & \gamma \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} e & \gamma \\ 0 & 1 \end{pmatrix}}_{n \text{ times}} \underbrace{\begin{pmatrix} e^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-1} & 0 \\ 0 & 1 \end{pmatrix}}_{n \text{ times}}.$$

So by (1.6.2), we have $\tau(g) \leq |a| + 2n$. By (1.6.3), we see that

$$\left(\frac{e^n - 1}{e - 1} \right) \gamma = e^{-a}b$$

so $e^n \leq e(e-1)(e^{-a}b) + 1$. We bound the word weight:

$$e^{\tau(g)} \leq e^{|a|+2n} \leq e^{|a|}(e(e-1)(e^{-a}b) + 1)^2 \leq (e(e-1))^2 \omega^2(g).$$

Since the word weight dominates every weight on G, it follows that ω is equivalent to the word weight on G.

We give some examples of nonsolvable Lie groups.

Example 1.6.4. Let $G = SL(n, \mathbb{R})$. We show that the word gauge τ is strongly equivalent to the Riemannian symmetric function σ. It suffices to show that σ strongly dominates τ. Let $g \in G$. Let $g = UP$ be the Cartan decomposition of g, and let V be a unitary matrix such that V^*PV is a positive diagonal matrix D. Then by [HW, (2.2ab)] we have

$$\sigma(g) = \sigma(P) = \sigma(D) = \| \log(D) \| = \max |\lambda_i|, \tag{1.6.5}$$

where

$$D = \begin{pmatrix} e^{\lambda_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{\lambda_n} \end{pmatrix}.$$

Let A be a generating set for G. Then if $\mathcal{U}$ denotes the set of unitary matrices, we have that

$$\mathcal{U}A\mathcal{U} \cap G$$

is a bounded subset of G and so contained in A^m for some m. Let k be the smallest integer greater than or equal to $\sigma(g)$. Then we have

$$\tau(g) = \tau(UVPV^*) \leq m\tau(P) \leq mk\tau(P^{1/k}) \leq mkC \leq mC(\sigma(g)+1), \tag{1.6.6}$$

where C is a constant which depends only on the norm of $P^{1/k}$, which is less than or equal to e by (1.6.5). Hence (1.6.6) holds for all $g \in G$ and σ strongly dominates τ.

It follows that $\sigma \sim_s \tau$ since τ strongly dominates every gauge on G. Thus if $\omega = e^\tau$ is the word weight, we have

$$\omega \sim e^\sigma. \tag{1.6.7}$$

Define another weight θ on G by $\theta(g) = \max(\| g \|, \| g^{-1} \|)$. We show that θ is also equivalent to ω. By (1.6.7), it suffices to show that θ dominates e^σ. We have

$$e^{\sigma(g)} = e^{\sigma(D)} = e^{\max|\lambda_i|} \leq \max(e^{\lambda_i}, e^{-\lambda_i}) = \max(\| D \|, \| D^{-1} \|) = \theta(D) \leq K\theta(g), \tag{1.6.8}$$

where K is any constant greater than or equal to the square of $\sup_{h \in \mathcal{U}} \theta(h)$. Since (1.6.8) holds for all $g \in G$, it follows that $\theta \sim \omega$.

Since ω dominates every weight on G, we know ω bounds Ad. We see this directly for the case $n = 2$. A calculation shows that if

$$g = \begin{pmatrix} E & F \\ G & H \end{pmatrix},$$

then

$$Ad_g = \begin{pmatrix} EH+FG & HG & -FE \\ 2HF & H^2 & -F^2 \\ -2GE & -G^2 & E^2 \end{pmatrix} \tag{1.6.9}$$

in some fixed basis for the Lie algebra. Then from the fact that $\omega \sim \theta$, it is easy to see that ω bounds Ad.

Example 1.6.10. Let $G = GL(n, \mathbb{R})$, and let ω be the word weight on G. Define $\theta(g) = \max(\| g \|, \| g^{-1} \|)$. We show that $\omega \sim \theta$. Let H be the subgroup of G of matrices with determinant ± 1. We have a natural isomorphism of groups

$$\mathbb{R} \times H \cong G, \tag{1.6.11}$$

given by $(r, [a_{ij}]) \mapsto [e^r a_{ij}]$. Since $H/SL(n, \mathbb{R}) \cong \mathbb{Z}_2$, an argument similar to the one used in Example 1.6.4 shows that $\theta \restriction_H$ is equivalent to the word weight e^{τ_H} on H. Similarly, $\theta \restriction_{\mathbb{R}}$ gives the weight $e^{|r|}$ on $\mathbb{R}$ (in the decomposition (1.6.11)), which is easily seen to be equivalent to the word weight $e^{\tau_{\mathbb{R}}}$ on $\mathbb{R}$. Note that

$$\theta(e^r [a_{ij}]) = \theta \restriction_{\mathbb{R}} (r) \quad \theta \restriction_H ([a_{ij}]).$$

It follows that θ dominates the product $e^{\tau_{\mathbb{R}} + \tau_H}$. So, to see that θ dominates ω, it suffices to show that the gauge on G defined by

$$\tau(e^r [a_{ij}]) = \tau_{\mathbb{R}}(r) + \tau_H([a_{ij}]). \tag{1.6.12}$$

strongly dominates the word gauge.

Let U and V be respective generating sets for $\mathbb{R}$ and H. Then $W = U \times V$ is a generating set for G. If $(e^r [a_{ij}]) \in W^k$, then $r \in U^k$ and $[a_{ij}] \in V^k$. Hence $\tau_G(e^r [a_{ij}]) \leq \max(\tau_{\mathbb{R}}(r), \tau_H([a_{ij}])) \leq \tau_{\mathbb{R}}(r) + \tau_H([a_{ij}])$. So we have shown that τ defined by (1.6.12) strongly dominates the word gauge τ_G. It follows that e^τ dominates e^{τ_G}, and hence that θ dominates the word weight $\omega = e^{\tau_G}$. Since G is compactly generated, the word weight dominates θ, and we have $\theta \sim \omega$.

§2 Smooth Crossed Products

In this section, A will be a Fréchet algebra with a continuous action of a Lie group G. We ignore any *-structure on A, and we do not assume that the action of G on A is differentiable or strongly continuous. If σ is a scale on G, then we shall define the (L^1) Schwartz functions $\mathcal{S}_1^\sigma(G, A)$ from G to A. If the action of G on A is tempered in an appropriate sense, and σ satisfies certain simple properties, we will see that $\mathcal{S}_1^\sigma(G, A)$ is a Fréchet algebra under convolution, which we will denote by $G \rtimes^\sigma A$.

To simplify notation, we shall make the assumption that $\sigma \geq 1$ throughout §2. We lose no generality, since any scale σ is equivalent to either of the scales $\max(1, \sigma)$ or $1 + \sigma$.

§2.1 Fréchet Space Valued Schwartz Functions

Definition 2.1.0. Let $\sigma \geq 1$ be any scale on a Lie group G, and let E be any Fréchet space. Assume σ is bounded on compact subsets of G. Let $\| \quad \|_m$ be a family of increasing seminorms giving the topology of E. We define the σ-rapidly vanishing L^1 Schwartz functions $\mathcal{S}_1^\sigma(G, E)$ from G to E to be the set of differentiable functions $\psi \colon G \to E$ such that the seminorms

$$\| \psi \|_{d,\gamma,m} = \int_G \sigma^d(g) \| X^\gamma \psi(g) \|_m \, dg \tag{2.1.1}$$

are finite for each $d, m \in \mathbb{N}$ and $\gamma \in \mathbb{N}^q$. (Here $X^\gamma \psi(g)$ is defined by formula (1.2.1) with action $\beta_h(\psi)(g) = \psi(h^{-1}g)$.) We topologize $\mathcal{S}_1^\sigma(G, E)$ by these seminorms. To simplify notation, we shall refer to $\mathcal{S}_1^\sigma(G, E)$ as $\mathcal{S}^\sigma(G, E)$. Note that $C_c^\infty(G, E) \subseteq \mathcal{S}^\sigma(G, E)$.

Before giving conditions for $\mathcal{S}^\sigma(G, E)$ to be a Fréchet space, we define another set of functions from G to E, the set of C^∞-vectors of which will be precisely $\mathcal{S}^\sigma(G, E)$. (For this discussion, G can be any locally compact group.) Let $\mathcal{F}_d$ denote the space of all functions $\psi \colon G \to E$ for which

$$\| \psi(g) \|_{d,m} = \int_G \sigma^d(g) \| \psi(g) \|_m \, dg < \infty \tag{2.1.2}$$

for all m, where $\int$ denotes the upper integral. (The upper integral of a positive function $f \colon G \to \mathbb{R}$ is the infimum of the integrals of the countably infinite linear combinations $\sum c_n \xi_n$, where $c_n \geq 0$, the functions ξ_n are characteristic functions of integrable subsets of G (where integrable means Borel measurable with finite Haar measure), and $\sum c_n \xi_n$ dominates f pointwise almost everywhere [Tr, p. 468, p. 99].) For each $m \in \mathbb{N}$, the expression (2.1.2) then defines a seminorm on $\mathcal{F}_d$. Define a step function s from G to E to be a finite linear combination of the form $\sum e_n \xi_n$, where $e_n \in E$ and ξ_n is the characteristic function of an integrable subset of G. Let $\mathcal{L}_d$ denote the closure of step functions in $\mathcal{F}_d$ in the topology given by the seminorms (2.1.2). Since $\sigma^{d+1} \geq \sigma^d$, the seminorm $\| \quad \|_{d+1,m}$ dominates $\| \quad \|_{d,m}$ for each m, and we have natural canonical continuous injections $\mathcal{L}_{d+1} \hookrightarrow \mathcal{L}_d$.

Let N_d denote the set of functions on which the seminorms (2.1.2) vanish for all m. This is clearly a closed subspace of $\mathcal{L}_d$. If $f \in N_d$, then $\int \| f(g) \|_m \sigma^d(g) dg = 0$ for every m. Hence, since $\sigma \geq 1$, the function f vanishes almost everywhere. So for any $k \in \mathbb{N}$, we have $\int \| f(g) \|_m \sigma^k(g) dg = 0$ for every m, and $f \in N_k$. It follows that the sets N_d are independent of d.

We let L_d denote the quotient $\mathcal{L}_d / N_d$. Every element of L_d may then be thought of as an equivalence class of measurable functions. By the Fisher-Riesz theorem for Fréchet spaces,

the space L_d is complete for the topology given by the seminorms (2.1.2) [Tr, p. 468]. Since $N_{d+1} = N_d$, the map $\mathcal{L}_{d+1} \hookrightarrow \mathcal{L}_d$ gives a continuous injection

$$L_{d+1} \hookrightarrow L_d.$$

We define the *σ-rapidly vanishing L^1 functions from G to E* to be the intersection (or projective limit)

$$L_1^\sigma(G,E) = \cap_{d=0}^{\infty} L_d. \tag{2.1.3}$$

The topology on $L_1^\sigma(G,E)$ is easily seen to be given by the seminorms (2.1.2), where d ranges over all natural numbers. It is also easy to see from the definition (2.1.3) and the completeness of each L_d that $L_1^\sigma(G,E)$ is complete for this topology. To simplify notation, we will denote $L_1^\sigma(G,E)$ by $L^\sigma(G,E)$. We could have defined $L^\sigma(G,E)$ to be the closure of integrable step functions or of continuous functions with compact support in an appropriate space $\mathcal{F}_\infty$, and then moded out by the set of functions which lie in the kernels of all the seminorms (2.1.2) [Schw, §5]. An easy argument shows that we would have gotten the same space. Note that $L^\sigma(G,E) \hookrightarrow L^\tau(G,F)$ with continuous inclusion map if $E \hookrightarrow F$ with continuous inclusion, and σ dominates τ.

The action

$$(gF)(h) = F(g^{-1}h), \qquad g,h \in G \tag{2.1.4}$$

of G on $L^\sigma(G,E)$ is strongly continuous if σ is uniformly translationally equivalent (see (1.2.10)). To see this, let $C_c(G,E)$ denote the continuous functions with compact support from G to E. Then (2.1.4) gives an action of G on $C_c(G,E)$ which is easily seen to be strongly continuous for the inductive limit topology. Using the uniform translational equivalence of σ, the density of $C_c(G,E)$ in $L^\sigma(G,E)$, and the fact that the seminorms (2.1.2) are continuous for the inductive limit topology on $C_c(G,E)$, a quick calculation shows that G acts strongly continuously on $L^\sigma(G,E)$.

Theorem 2.1.5. *Let σ be a translationally equivalent scale on a Lie group G (see Definition 1.2.7). Then the space $\mathcal{S}_1^\sigma(G,E)$ is the set of C^∞-vectors for the action (2.1.4) of G on $L_1^\sigma(G,E)$. Hence $\mathcal{S}_1^\sigma(G,E)$ is complete and a Fréchet space. The set of compactly supported functions $C_c^\infty(G,E)$ is dense in $\mathcal{S}_1^\sigma(G,E)$. If $\sigma_1 \sim \sigma_2$, then $\mathcal{S}_1^{\sigma_1}(G,E) = \mathcal{S}_1^{\sigma_2}(G,E)$.*

Proof. Recall from Theorem 1.2.11 that σ is uniformly translationally equivalent since it is translationally equivalent. Let $\mathcal{S}$ denote $\mathcal{S}_1^\sigma(G,E)$, let L denote $L_1^\sigma(G,E)$, and let L^∞ denote

the set of C^∞-vectors for the action of G on L. A calculation similar to (1.2.9) shows that G acts strongly continuously and differentiably on $\mathcal{S}$.

We show that $L^\infty \subseteq S$. By convention, $\int \varphi(h)dh = \int \Delta(h)\varphi(h^{-1})dh$, where Δ is the modular function. By [DM, Thm 3.3], we may write any element of L^∞ as a finite sum of functions

(2.1.6)

$$\begin{aligned} f * \psi(g) &= \int_G f(h)\psi(h^{-1}g)dh = \int_G f(gh)\psi(h^{-1})dh \\ &= \int_G \Delta(h) f(gh^{-1})\psi(h)dh = \Delta(g)\int_G \Delta^{-1} f(gh^{-1})\psi(h)dh, \end{aligned}$$

where $f \in C_c^\infty(G)$, $\psi \in L^\infty$. It suffices to prove that functions of the form (2.1.6) lie in $\mathcal{S}$. We first show that such functions are continuous. Let $g_n \longrightarrow g_0$ in G, and let $\tilde{f} = \Delta^{-1}f$, $\tilde{f}_g(h) = \tilde{f}(g^{-1}h)$. Since Δ is continuous, it suffices to show that $g \mapsto \Delta^{-1}(g)(f*\psi)(g)$ is continuous. For any seminorm $\| \quad \|_d$ on E,

$$\begin{aligned} \| \Delta^{-1}(f*\psi)(g_n) - \Delta^{-1}(f*\psi)(g_0) \|_d &\le \int \| \tilde{f}(g_n h^{-1}) - \tilde{f}(g_0 h^{-1})\psi(h) \|_d \, dh \\ &\le \int \| \psi(h) \|_d \, dh \, \| \tilde{f}_{g_n^{-1}} - \tilde{f}_{g_0^{-1}} \|_\infty \\ &\le C_f \, \| \psi \|_{0,d} \| \tilde{f}_{g_n^{-1}} - \tilde{f}_{g_0^{-1}} \|_\infty \, . \end{aligned}$$

The last expression tends to zero as g_n tends to g_0, since $\tilde{f} \in C_c^\infty(G)$. Thus $f * \psi$ is a continuous function. Similar arguments, bringing derivatives inside the integral (2.1.6) to act on f, show that $f * \psi$ is a differentiable function from G to E. Thus L^∞ consists of differentiable functions from G to E. (An argument similar to [Co-Gr, Lemma 2.3.2] may also work for this.) Since the seminorms on L^∞ are precisely those on $\mathcal{S}$, we have $L^\infty \subseteq \mathcal{S}$.

To see that $C_c^\infty(G,E)$ is dense in $\mathcal{S}$, we apply the following lemma and the fact that $C_c^\infty(G,E)$ is dense in L. (Alternatively, one could probably use approximate units.)

Lemma 2.1.7. *Let E be a Fréchet space on which a Lie group G acts strongly continuously. Assume that F is a dense G-invariant subspace of E. Then the algebraic span $C_c^\infty(G)F$ is dense in E^∞.*

Proof. By [DM, Thm 3.3], every element of E^∞ is a finite sum of elements of the form $\alpha_\varphi(e) = \int \varphi(h)\alpha_h(e)dh$, where $e \in E$ and $\varphi \in C_c^\infty(G)$. Let $f_n \to e$ in E. Then $\alpha_\varphi(f_n)$ tends to $\alpha_\varphi(e)$ in E. $\square$

The last statement about equivalent scales is a simple calculation. This proves Theorem 2.1.5. $\square$

§2.2 Conditions for an Algebra

We give conditions on the scale σ and the action of G on a Fréchet algebra A which makes $\mathcal{S}^\sigma(G, A)$ an algebra under convolution. We begin with a general lemma for actions of G on Fréchet spaces.

Lemma 2.2.1. *Assume that the action α of G on a Fréchet space E is differentiable. Let $e \in E$. Let $\gamma \in \mathbb{N}^q$, where q is the dimension of the Lie algebra $\mathfrak{G}$. Assume that $\sigma \geq 1$ and σ_- bounds Ad (recall $\sigma_-(g) = \sigma(g^{-1})$). Then there exists a constant $D > 0$ and $p \in \mathbb{N}$ such that*

$$\| T X^\gamma(\alpha_h(e)) \|_d \leq D\sigma^p(h) \sum_{\beta \leq \gamma} \| T\alpha_h(X^\beta e) \|_d, \tag{2.2.2}$$

for all $d \in \mathbb{N}$, $h \in G$, $e \in E$, and all continuous linear maps $T \colon E \to E$.

Here $X^\gamma \psi$ is defined by formula (1.2.1) with $\beta = \alpha$.

Proof. Consider the case $\gamma_i = (0, \dots 1, \dots 0)$, where the 1 is in the ith spot. By the chain rule,

$$\begin{aligned} X^{\gamma_i} \alpha_h(e) &= \frac{d}{dt} \alpha_{exp(tX_i)h}(e) \restriction_{t=0} \quad = \frac{d}{dt} \alpha_{hexp(Ad_{h^{-1}}(tX_i))}(e) \restriction_{t=0} \\ &= \sum_{j=1}^q \alpha_h(X^{\gamma_j} e) \frac{d}{dt} (Ad_{h^{-1}}(tX_i))^{(j)}|_{t=0} = \sum_{j=1}^q \alpha_h(X^{\gamma_j} e)(Ad_{h^{-1}})_{ji}. \end{aligned} \tag{2.2.3}$$

(Here if $Y \in \mathfrak{G}$, then the $Y^{(j)}$'s are the coordinates of Y in the basis $X_1, \dots X_q$.) Since σ_- bounds Ad, the matrix elements $(Ad_{h^{-1}})_{ji}$ have their absolute values bounded by $D\sigma^p(h)$ for some constant D and $p \in \mathbb{N}$. Thus

$$\| T X^{\gamma_i}(\alpha_h(e)) \|_d \leq D\sigma^p(h) \sum_{j=1}^q \| T\alpha_h(X^{\gamma_j} e) \|_d \, .$$

The case of general $\gamma \in \mathbb{N}^q$ follows by repeated application of X^{γ_i}'s to Eqn 2.2.3, and again using the bound on $|(Ad_{h^{-1}})_{ji}|$ by $D\sigma^p(h)$. This proves Lemma 2.2.1. $\square$

Definition 2.2.4. Let $\| \quad \|_m$ be an increasing sequence of seminorms on a Fréchet algebra A giving the topology of A. We let $\mathrm{Aut}(A)$ denote the group of continuous algebra automorphisms of A. We say that a group homomorphism $\alpha \colon G \to \mathrm{Aut}(A)$ is a *continuous action* of G on A if $g \mapsto \| \alpha_g(a) \|_m$ is continuous for each $m \in \mathbb{N}$ and $a \in A$. From now on, we shall

only consider continuous actions of G. Let $\sigma \geq 1$ be a scale on G. We say that the action of G on A is *σ-tempered* if for every m there are k, l, and C so that

$$\| \alpha_g(a) \|_m \leq C\sigma^k(g) \| a \|_l, \qquad a \in A,\ g \in G. \tag{2.2.5}$$

If the scale is understood, we simply say that the action is tempered. This definition is analogous to the definition of smoothness in [ENN] for an action of $\mathbb{R}$ (without the differentiability condition), and the notion of a tempered G-module for a nilpotent Lie group in [DuC 1, §4], [DuC 3] [DuC 2].

We remark that tempered does not imply strongly continuous. For example, let $\mathbb{R}$ act by translation on the bounded continuous functions $C_b(\mathbb{R})$ with sup norm topology.

Theorem 2.2.6. *Let σ be a sub-polynomial scale on a locally compact group G. Let α be a σ-tempered action of G on a Fréchet algebra A. Then the Fréchet space $L_1^\sigma(G, A)$ is a Fréchet algebra under convolution.*

Assume in addition that G is a Lie group, and that either σ_- bounds Ad or that G acts differentiably on A. Then the Fréchet space $\mathcal{S}_1^\sigma(G, A)$ is a Fréchet algebra under convolution.

When the conditions of Theorem 2.2.6 are satisfied, we define the *smooth crossed product of A by (G, σ)* to be the Fréchet space $\mathcal{S}_1^\sigma(G, A)$ with convolution multiplication. We denote it by $G \rtimes^\sigma A$, or $G \rtimes A$ if the scale σ is understood.

Proof. We check that $\mathcal{S}^\sigma(G, A)$ is closed under multiplication. Without loss of generality, we assume $\sigma \geq 1$. Assume that σ_- bounds Ad, so that Lemma 2.2.1 applies. We compute the

seminorm of a product in $\mathcal{S}^\sigma(G, A)$.

$$
\begin{aligned}
&\| \psi * \varphi \|_{m,\gamma,d} = \int_G \sigma^m(g) \| X^\gamma(\psi * \varphi)(g) \|_d \, dg \\
&= \int \sigma^m(g) \| \int \psi(h)\alpha_h((X^\gamma \varphi_h)(g))dh \|_d \, dg, \quad \text{def of conv} \\
&\le \int\int \sigma^m(g) \| \psi(h) \|_{d'} \| \alpha_h((X^\gamma \varphi_h)(g)) \|_{d'} \, dhdg, \quad A \text{ Fréchet} \\
&\le \int\int \sigma^m(g)\sigma^k(h) C \| \psi(h) \|_{d'} \| (X^\gamma \varphi_h)(g) \|_l \, dhdg, \quad \text{tempered action} \\
&\le \sum_{\beta \le \gamma} \int\int \sigma^m(g) \| \sigma^k \psi(h) \|_{d'} \, CD\sigma^p(h) \| (X^\beta \varphi)_h(g) \|_l \, dhdg, \quad (2.2.2) \qquad (2.2.7) \\
&\le \tilde{C} \sum_{\beta \le \gamma} \int\int \| \sigma^{k+p+mr}\psi(h) \|_{d'} \| \sigma^{mr}(X^\beta \varphi)(h^{-1}g) \|_l \, dgdh, \ \text{sub-poly} \\
&= \tilde{C} \sum_{\beta \le \gamma} \int\int \| \sigma^{k+p+mr}\psi(h) \|_{d'} \| \sigma^{mr}(X^\beta \varphi)(g) \|_l \, dgdh, \\
&= \tilde{C} \sum_{\beta \le \gamma} \| \psi \|_{k+p+mr,0,d'} \| \varphi \|_{mr,\beta,l} \qquad \text{def of norms.}
\end{aligned}
$$

So $\mathcal{S}^\sigma(G, A)$ is a Fréchet algebra. The computation (2.2.7) without the derivatives shows that $L^\sigma(G, A)$ is a Fréchet algebra (without assuming that σ_- bounds Ad). If G acts differentiably on A, the estimates [Sc 3, (6.23),(6.24)] show that $\mathcal{S}^\sigma(G, A)$ is a Fréchet algebra under convolution. (Alternatively, one could do an appropriate change of variables for h in the early stages of (2.2.7).) This proves Theorem 2.2.6. □

See §5 for examples. We prove some elementary results on ideals and quotients, which we will be using in a later paper.

Proposition 2.2.8. *Let G, σ and A be as in Theorem 2.2.6. Let I be any closed G-invariant two-sided ideal of A. Let J denote the closed subspace of $G \rtimes A$ of functions which take their values in I. Then J is a two-sided ideal in $G \rtimes A$ and $G \rtimes A/J \cong G \rtimes (A/I)$.*

Proof. First note that if the action of G on A is tempered, then the action of G on A/I is tempered. Hence we may form the crossed product $G \rtimes (A/I)$. Let $\pi: G \rtimes A \to G \rtimes (A/I)$ be the canonical map. We show that π is onto. The map $L^\sigma(G, A) \to L^\sigma(G, A/I)$ is onto since the projective tensor product of surjective maps is surjective[Schw, §5][Tr, Prop 43.9]. By Theorem A.8, the corresponding map of C^∞-vectors for the action of G be left translation is surjective, so by Theorem 2.1.5, π is surjective.

Clearly the kernel of π is J. □

§3 m-CONVEXITY

§3.1 CONDITIONS FOR m-CONVEXITY OF THE CROSSED PRODUCT

Assume that A is an m-convex Fréchet algebra. That is, assume that the seminorms topologizing A can be taken to be submultiplicative. We give conditions so that the convolution algebra $G \rtimes A$ is m-convex.

Definition 3.1.1. Let $\sigma \geq 1$ be a scale on a topological group G. We say that the action of G on A is *m-σ-tempered* if there exists a family of submultiplicative seminorms $\{\| \quad \|_m\}$ topologizing A such that for every $m \in \mathbb{N}$, there exists $C > 0$ and $d \in \mathbb{N}$, for which

$$\| \alpha_g(a) \|_m \leq C\sigma^d(g) \| a \|_m, \qquad a \in A, g \in G. \tag{3.1.2}$$

Note that this is similar to the definition of a σ-tempered action (2.2.4), except that we have submultiplicative seminorms, and the same seminorm appears both on the left and the right of the inequality (3.1.2). We call the seminorms $\| \quad \|_m$ *(G, σ)-stable submultiplicative* seminorms for A. If the scale σ is understood, we simply say that the action is m-tempered. If the action of G is m-tempered, it is clearly tempered. This more restrictive notion of an m-tempered action seems to be necessary in the proof of Theorem 3.1.7 below, to obtain the m-convexity of the crossed product.

Question 3.1.3. Is a tempered action of a group on an m-convex Fréchet algebra always m-tempered ?

The reverse direction of the following theorem follows from [MRZ, Lemma 2]. The forward direction is an easy exercise. See [Pe, Thm 3.2] for a similar result for matrix algebras.

Theorem 3.1.4. *A Fréchet algebra $\mathcal{A}$ is m-convex if and only if for every family (or equivalently for any one family) $\{\| \quad \|_m\}$ of increasing seminorms giving the topology of $\mathcal{A}$, and for every $p \in \mathbb{N}$, there exists $C > 0$, $r \geq p$ such that*

$$\| a_1 \dots a_n \|_p \leq C^n \| a_1 \|_r \dots \| a_n \|_r, \tag{3.1.5}$$

for all n-tuples $a_1 \dots a_n$ of elements of A, and all $n \in \mathbb{N}$. $\square$

Let $\mathcal{A}_m$ be a family of m-convex Fréchet algebras with continuous algebra homomorphisms $\pi_{mk} : \mathcal{A}_m \to \mathcal{A}_k$ for $m \geq k$. Assume that $\pi_{ml} \circ \pi_{lk} = \pi_{mk}$ for $k \leq l \leq m$, $\pi_{mm} = \mathrm{id}$

and that $\mathcal{A}_{m+1}$ has dense image in $\mathcal{A}_m$. Let $\prod_{m=0}^{\infty} \mathcal{A}_m$ be the topological cartesian product of the spaces $\mathcal{A}_m$. We call the subset of $\prod_{m=0}^{\infty} \mathcal{A}_m$ of tuples $(a_0, a_1, \dots)$ which satisfy $\pi_{m+1m}(a_{m+1}) = a_m$ for all m the *projective limit* of the $\mathcal{A}_m$'s, denoted by $\mathcal{A} = \varprojlim \mathcal{A}_m$. The image of $\mathcal{A}$ is dense in each $\mathcal{A}_m$, and is complete for the strongest topology which dominates the topology on each $\mathcal{A}_m$ (or, in other words $\mathcal{A}$ is a Fréchet space for the topology given by all the seminorms on all the $\mathcal{A}_m$'s) [Mi] [Ph, 1.4-5]. Since we may take every seminorm submultiplicative, $\mathcal{A}$ is also an m-convex Fréchet algebra. Similarly, if $\mathcal{A}$ is any m-convex Fréchet algebra, and $\| \quad \|_m$ is a family of increasing submultiplicative seminorms giving the topology for $\mathcal{A}$, then $\mathcal{A}$ is the projective limit $\varprojlim \mathcal{A}_m$ of the Banach algebra completions $\mathcal{A}_m$ in $\| \quad \|_m$[Mi]. We say that a scale $\sigma \geq 1$ is *m-sub-polynomial* if there exists $C > 1$ and $l \in \mathbb{N}$ such that

$$\sigma(g_1 \dots g_n) \leq C^n \sigma^l(g_1) \dots \sigma^l(g_n), \tag{3.1.6}$$

for all $g_1, \dots g_n \in G$ and $n \in \mathbb{N}$. If $\sigma_1 \sim \sigma_2$, then σ_1 is m-sub-polynomial iff σ_2 is. Any scale which is equivalent to a weight or a gauge is m-sub-polynomial. Recall that $\sigma_-(g) = \sigma(g^{-1})$.

Theorem 3.1.7. *Let σ be an m-sub-polynomial scale on a Lie group G such that either σ_- bounds Ad (for example, σ could be any weight or gauge that bounds Ad) or G acts differentiably on A, and let the action of G on an m-convex Fréchet algebra A be m-σ-tempered. Then the convolution algebra $G \rtimes^\sigma A$ is an m-convex Fréchet algebra. In fact $G \rtimes^\sigma A$ is the projective limit of Fréchet algebras $G \rtimes^\sigma A_m$, where A_m is the completion of A in the mth (G, σ)-stable submultiplicative seminorm $\| \quad \|_m$.*

Similarly, if G is any locally compact group, then the corresponding conclusions are true for the Fréchet algebra $L_1^\sigma(G, A)$, using only the assumptions that the action of G on A is m-σ-tempered, and that σ is m-sub-polynomial.

Proof. First consider the case where A is a Banach algebra. Without loss of generality, assume $\sigma \geq 1$. Let $C_1 > 1$ and $l \in \mathbb{N}$ be such that

$$\sigma(g_1 \dots g_n) \leq C_1^n \sigma^l(g_1) \dots \sigma^l(g_n), \qquad g_1, \dots g_n \in G.$$

Let $\| \quad \|$ be a (G, σ)-stable submultiplicative seminorm giving the topology for A, and let $C > 1$ and $m \in \mathbb{N}$ be such that $\| \alpha_g(a) \| \leq C\sigma^m(g) \| a \|$. We verify the m-convexity by showing that the family of increasing seminorms

$$\| \psi \|_p' = \max \| \psi \|_{p,\gamma} = \max \int \| \sigma^p X^\gamma \psi(g) \| \, dg \tag{3.1.8}$$

satisfies (3.1.5).

To prepare to estimate $\| \psi_1 * \ldots \psi_n \|_{d,\gamma}$, we write $\psi_1 * \ldots \psi_n(g)$ as

$$\int \cdots \int \alpha_{\eta_1}(\psi_1(h_1)) \ldots \alpha_{\eta_{n-1}}(\psi_{n-1}(h_{n-1}))\alpha_{\eta_n}(\psi_n(h_n)) dh_1 \ldots dh_{n-1} \tag{3.1.9}$$

where $h_1, \ldots, h_{n-1}$ are the variables of integration, $h_n = h_{n-1}^{-1} \ldots h_1^{-1} g$, $\eta_1 = e$, $\eta_k = h_1 \ldots h_{k-1}$. We proceed to estimate. Using (3.1.9) and the left invariance of Haar measure,

$$\begin{aligned} \| \psi_1 * \ldots \psi_n \|_{d,\gamma} &= \int \sigma^d(g) \| (X^\gamma(\psi_1 \ldots \psi_n))(g) \| dg \\ &\le \int \cdots \int \sigma^d(g) \| \alpha_{\eta_1}(\psi_1(h_1)) \\ &\quad \ldots \alpha_{\eta_{n-1}}(\psi_{n-1}(h_{n-1}))\alpha_{\eta_n}(X^\gamma(\psi_n)_{\eta_n}(g)) \| dh_1 \ldots dh_{n-1} dh_n. \end{aligned} \tag{3.1.10}$$

By Lemma 2.2.1, for some $j, C_2 > 1$ depending only on our q-tuple γ, the integrand in (3.1.10) is bounded by

$$C_2 \sigma^d(g) \sigma^j(\eta_n) \sum_{\beta \le \gamma} \| \alpha_{\eta_1}(\psi_1(h_1)) \ldots \alpha_{\eta_{n-1}}(\psi_{n-1}(h_{n-1}))\alpha_{\eta_n}(X^\beta \psi_n)(h_n) \| . \tag{3.1.11}$$

Since the action of G is m-σ-tempered, we may bound the summand in (3.1.11).

$$\begin{aligned} &\| \alpha_{\eta_1}(\psi_1(h_1)) \ldots \alpha_{\eta_n}(X^\beta \psi_n)(h_n) \| \\ &\quad \le \| \psi_1(h_1) \| \| \alpha_{h_1}(\psi_2)(h_2) \ldots \alpha_{\eta_n}(X^\beta \psi_n)(h_n) \| \quad \text{since } \eta_1 = e,\ \eta_2 = h_1 \\ &\quad \le \| \psi_1(h_1) \| \ C\sigma^m(h_1) \ \| \psi_2(h_2) \| \| \alpha_{h_1^{-1}\eta_2}(\psi_3)(h_3) \ldots \alpha_{h_1^{-1}\eta_n}(X^\beta \psi_n)(h_n) \| \\ &\quad \cdots \qquad \cdots \qquad \cdots \\ &\quad \le C^{n-1} \sigma^m(h_1) \ldots \sigma^m(h_{n-1}) \ \| \psi_1(h_1) \| \| \psi_2(h_2) \| \ldots \| X^\beta \psi_n(h_n) \| \end{aligned} \tag{3.1.12}$$

where we have repeatedly used the (G, σ)-stable submultiplicative property of our seminorm $\| \ \ \|$. Since σ is m-sub-polynomial,

$$\sigma^j(\eta_n) \le C_1^{jn} \sigma^{jl}(h_1) \ldots \sigma^{jl}(h_{n-1}) \tag{3.1.13}$$

and

$$\sigma^d(g) \le C_1^{dn} \sigma^{dl}(h_1) \ldots \sigma^{dl}(h_n). \tag{3.1.14}$$

Now we are ready to plug everything back in and make our estimate. Plugging (3.1.12) and (3.1.13) into (3.1.11), we see that the integrand of (3.1.10) is bounded by

$$\begin{aligned} &\sigma^d(g) \sum_{\beta \le \gamma} C_2 (CC_1^j)^n \| \sigma^{jl+m} \psi_1(h_1) \| \\ &\qquad \ldots \| \sigma^{jl+m} \psi_{n-1}(h_{n-1}) \| \| (X^\beta \psi_n)(h_n) \| \end{aligned} \tag{3.1.15}$$

Using (3.1.14), we see that (3.1.15) is bounded by

$$\sum_{\beta\leq\gamma} C_2(CC_1^{j+d})^n \parallel \sigma^{jl+m+dl}\psi_1(h_1) \parallel \ldots \parallel \sigma^{dl}(X^\beta\psi_n)(h_n) \parallel . \tag{3.1.16}$$

Let $t = jl + m + dl$. Then plugging (3.1.16) back into the integral (3.1.10), we see that

$$\begin{aligned} &\parallel \psi_1 \ldots \psi_n \parallel_{d,\gamma} \\ &\leq \sum_{\beta\leq\gamma} C_2(CC_1^{j+d})^n \parallel \psi_1 \parallel_{t,0} \ldots \parallel \psi_{n-1} \parallel_{t,0} \parallel \psi_n \parallel_{t,\beta} \\ &\leq C_3^n \parallel \psi_1 \parallel'_p \ldots \parallel \psi_{n-1} \parallel'_p \parallel \psi_n \parallel'_p \end{aligned} \tag{3.1.17}$$

where $p \geq t, |\gamma|$, and C_3 is a sufficiently large constant. By Theorem 3.1.4, this proves the m-convexity of $G \rtimes^\sigma A$.

Now let A be any m-convex Fréchet algebra, and let $\parallel \ \ \parallel_m$ be a family of increasing (G, σ)-stable submultiplicative seminorms for the topology of A. Let A_m denote the completion of A in $\parallel \ \ \parallel_m$. Then $G \rtimes^\sigma A_m$ is m-convex by what we have just shown. Since $G \rtimes^\sigma A$ is precisely the set of differentiable functions for which each of the seminorms for $G \rtimes^\sigma A_m$ is finite for each m (see (2.1.1)), $G \rtimes^\sigma A$ can be identified with the projective limit $\varprojlim G \rtimes^\sigma A_m$, and so $G \rtimes^\sigma A$ is an m-convex Fréchet algebra.

The same proof without derivatives gives the corresponding results for $L_1^\sigma(G, A)$. $\square$

Just as the notion of m-convexity has an equivalent condition in terms of an arbitrary increasing family of seminorms on A (see Theorem 3.1.4), so does the notion of an m-tempered action. We prove this for the case when σ is a weight. The case of a gauge σ may be obtained by applying the theorem to the weight $1 + \sigma$.

Theorem 3.1.18. *An action of a group G with weight σ on a Fréchet algebra A is m-σ-tempered if and only if for every family (or equivalently for any one family) $\{\parallel \ \ \parallel_m\}$ of increasing seminorms for A we have that for every $m \in \mathbb{N}$, there exists $C > 0$, $d, r \geq m$ such that*

(3.1.19)

$$\parallel \alpha_{g_1}(a_1) \ldots \alpha_{g_n}(a_n) \parallel_m \leq C^n \Big(\sigma(g_1)\sigma(g_1^{-1}g_2) \ldots \sigma(g_{n-1}^{-1}g_n)\sigma(g_n) \Big)^d \parallel a_1 \parallel_r \ldots \parallel a_n \parallel_r,$$

for all n-tuples $a_1, \ldots, a_n \in A$, $g_1, \ldots, g_n \in G$, and all $n \in \mathbb{N}$.

Proof. First assume that the action of G on A is m-σ-tempered. Then we have

$$\begin{aligned}
\| \alpha_{g_1}(a_1)\dots\alpha_{g_n}(a_n) \|_m &\leq C\sigma^d(g_1) \| a_1 \|_m \| \alpha_{g_1^{-1}g_2}(a_2)\dots\alpha_{g_1^{-1}g_n}(a_n) \|_m \\
&\leq C^2\sigma^d(g_1)\sigma^d(g_1^{-1}g_2) \| a_1 \|_m \| a_2 \|_m \| \alpha_{g_2^{-1}g_3}(a_3)\dots\alpha_{g_2^{-1}g_n}(a_n) \|_m \\
&\dots \qquad \dots \\
&\leq C^n\sigma^d(g_1)\sigma^d(g_1^{-1}g_2)\dots\sigma^d(g_{n-1}^{-1}g_n) \| a_1 \|_m \dots \| a_n \|_m
\end{aligned} \tag{3.1.20}$$

for any (G,σ)-stable submultiplicative seminorms for A. It follows easily that if $\| \quad \|_m'$ is any equivalent family of seminorms, then (3.1.19) is satisfied for $\| \quad \|_m'$ with possibly different constants C and d than were used in (3.1.2). (In fact, it is easy to check that if one family of seminorms satisfies (3.1.19), then every increasing one does.)

To show the converse, let $\| \quad \|_m$ be any increasing family of seminorms which satisfy (3.1.19). For $r \in \mathbb{N}$, let m, C and d be as in (3.1.19). Let $U_m = \{\, a \in A \mid \| a \|_m \leq 1 \,\}$. Then by (3.1.19),

$$\alpha_{g_1}(U_r)\dots\alpha_{g_n}(U_r) \subseteq C^n\Big(\sigma(g_1)\sigma(g_1^{-1}g_2)\dots\sigma^d(g_{n-1}^{-1}g_n)\sigma(g_n)\Big)^d U_m \tag{3.1.21}$$

for all n-tuples $g_1,\dots g_n \in G$. For $g_1,\dots g_n \in G$, define the balanced neighborhoods of zero

$$W_{g_1,\dots g_n} = \frac{1}{C^n(\sigma(g_1)\sigma(g_1^{-1}g_2)\dots\sigma(g_{n-1}^{-1}g_n)\sigma(g_n))^d}\Big(\alpha_{g_1}(U_r)\dots\alpha_{g_n}(U_r)\Big). \tag{3.1.22}$$

Let

$$W_n = \bigcup_{g_1,\dots g_k \in G, \quad k \leq n} \Big(W_{g_1,\dots g_k}\Big).$$

Then $W_{n+1} \supseteq W_n$. Define decreasing convex neighborhoods of zero V_r by

$$V_r = \bigcup_{n=1}^{\infty} conv(W_n) \tag{3.1.23}$$

where $conv(S)$ denotes the smallest convex subset of A which contains a set S. Then $V_r \subseteq U_m$ so the V_r form a basis of zero neighborhoods for the topology of A. Also, the V_r are convex and balanced. (V_r is convex since it is the union of increasing convex sets.)

We show that $W_nW_m \subseteq W_{n+m}$. Let $k \leq n$, $l \leq m$. It suffices to show that $W_{g_1,\dots g_k}W_{h_1,\dots h_l} \subseteq W_{g_1,\dots g_k,h_1,\dots h_l}$. Let $w_1 = \alpha_{g_1}(u_1)\dots\alpha_{g_k}(u_k)/C^k(\sigma(g_1)\sigma(g_1^{-1}g_2)\dots\sigma(g_k))^d \in W_{g_1,\dots g_k}$ and $w_2 = \alpha_{h_1}(\tilde{u}_1)\dots\alpha_{h_l}(\tilde{u}_l)/C^p(\sigma(h_1)\sigma(h_1^{-1}h_2)\dots\sigma(h_l))^d \in W_{h_1,\dots h_l}$. Then

(3.1.24)

$$\frac{1}{C^{k+l}(\sigma(g_1)\sigma(g_1^{-1}g_2)\dots\sigma(g_k)\sigma(h_1)\sigma(h_1^{-1}h_2)\dots\sigma(h_l))^d}$$

$$\leq \frac{1}{C^{k+l}(\dots)},$$

since $\sigma(g_k^{-1}h_1) \leq \sigma(g_k)\sigma(h_1)$. So $w_1 w_2 \in W_{g_1,\ldots g_k,h_1,\ldots h_l}$. It follows that $V_r^2 \subseteq V_r$.

Also, from the submultiplicativity of σ, we see that $\alpha_g(W_{g_1,\ldots g_n}) \subseteq \sigma^{2d}(g)W_{gg_1,\ldots gg_n}$. It follows that $\alpha_g(W_n) \subseteq \sigma^{2d}(g)W_n$ and $\alpha_g(V_r) \subseteq \sigma^{2d}(g)V_r$.

Define new seminorms by $\| a \|_r' = \inf\{ c > 0 \,|\, a \in cV_r\}$. Then $\| ab \|_r' \leq \| a \|_r' \| b \|_r'$ and $\| \alpha_g(a) \|_r' \leq \sigma^{2d} \| a \|_r'$. Hence condition (3.1.19) implies that the action of G is m-σ-tempered. □

Question 3.1.25. Let σ be a weight on G. Are there examples of m-convex Fréchet algebras A with σ-tempered (not m-σ-tempered) actions of G for which the crossed product $G \rtimes^\sigma A$ is not m-convex ?

Example 3.1.26. In a preliminary version of this paper [Sc 1, chap 1, §3], I had a different condition than m-temperedness to prove the m-convexity of the crossed product. Let $\sigma \geq 1$ be a weight on G. We say that an action of G on A is *strongly m-σ-tempered* if for every family (or equivalently for any one family) $\{\| \;\; \|_m\}$ of increasing seminorms for A we have that for every $m \in \mathbb{N}$, there exists $C > 0$, $r \geq m$, $d \in \mathbb{N}$ such that

$$(3.1.27) \quad \| \alpha_{g_1}(a_1)\ldots\alpha_{g_n}(a_n) \|_m \leq C^n \left(\max_{\tau_1+\ldots\tau_n \leq d} \sigma^{\tau_1}(g_1)\ldots\sigma^{\tau_n}(g_n) \right) \| a_1 \|_r \ldots \| a_n \|_r,$$

for all n-tuples $a_1 \ldots a_n \in A$, $g_1 \ldots g_n \in G$ and all $n \in \mathbb{N}$. (Note that r and d do not depend on n.)

This condition implies m-temperedness since if $\tau_{i_1}, \ldots \tau_{i_d}$ are the only nonzero τ's on the right of the inequality (3.1.27), we have

$$\begin{aligned}
&\sigma^{\tau_1}(g_1)\ldots\sigma^{\tau_n}(g_n) \leq \sigma^d(g_{i_1})\ldots\sigma^d(g_{i_d}) \\
&\quad \leq \Big((\sigma(g_{i_1}^{-1}g_{i_1+1})\ldots\sigma(g_{n-1}^{-1}g_n)\sigma(g_n))\ldots(\sigma(g_{i_d}^{-1}g_{i_d+1})\ldots\sigma(g_{n-1}^{-1}g_n)\sigma(g_n)) \Big)^d \\
&\quad \leq \Big(\sigma(g_1^{-1}g_2)\ldots\sigma(g_{n-1}^{-1}g_n)\sigma(g_n) \Big)^{d^2}
\end{aligned}$$

by the submultiplicative property of the weight σ.

We give an example of an m-tempered action which is not strongly m-tempered. Let $\mathbb{Q}^\infty$ be the direct sum

$$\bigoplus_{n=0}^{\infty} \mathbb{Q}$$

of countably many copies of $\mathbb{Q}$ with pointwise addition. Let ω be the weight on $\mathbb{Q}^\infty$ defined by

$$\omega(\vec{r}) = \prod^{\infty}(1 + |r_i|).$$

(Note that if $\vec{r} \in \mathbb{Q}^\infty$, only finitely many r_i's are nonzero, so the definition of ω makes sense.) Let A be the commutative Banach algebra $l^1(\mathbb{Q}^\infty, \omega)$ of functions from $\mathbb{Q}^\infty$ to $\mathbb{C}$ which satisfy

$$\| \varphi \|_1 = \sum_{\mathbb{Q}^\infty} \omega(\vec{r})|\varphi(\vec{r})| < \infty, \tag{3.1.28}$$

with convolution multilplication. The norm $\| \quad \|_1$ is easily seen to be submultiplicative.

Let G be the discrete group consisting of tuples $(q_1, \ldots, q_n, 1, 1, \ldots)$ of infinite length, with only finitely many entries not equal to one, and each q_i a positive rational number. With pointwise multiplication, G is easily seen to be a group. (Note that G is not finitely generated.) Define a weight γ on G by

$$\gamma(\vec{q}) = \prod_{i=1}^{\infty} \max(q_i, 1/q_i).$$

Define an action α of G on A by

$$\alpha_{\vec{q}}(\varphi)(\vec{r}) = \varphi(\vec{q}\,.\,\vec{r}),$$

where $\vec{q}\,.\,\vec{r} = (q_1, \ldots q_n, 1, \ldots)\,.\,(r_1, \ldots r_n, 0, \ldots)$ is given by pointwise multiplication $(q_1 r_1, \ldots q_n r_n, 0, \ldots)$. Then $\alpha_{1/\vec{q}} = \alpha_{\vec{q}}^{-1}$, and it is not too hard to show that α gives an action of G on A by algebra automorphisms. We have

$$\| \alpha_{\vec{q}}(\varphi) \|_1 = \sum_{\vec{r} \in \mathbb{Q}^\infty} \omega(\vec{r})|\varphi(\vec{q}\,.\,\vec{r})| = \sum_{\vec{r} \in \mathbb{Q}^\infty} \omega(1/\vec{q}\,.\,\vec{r})|\varphi(\vec{r})| \le \gamma(\vec{q}) \| \varphi \|_1, \tag{3.1.29}$$

since $(1 + |r_i/q_i|) \le \max(q_i, 1/q_i)(1 + |r_i|)$. Hence $\alpha_{\vec{q}}$ is a continuous automorphism of A. In fact, by (3.1.29), the submultiplicative norm $\| \quad \|_1$ on A is (G, γ)-stable. Hence the action of G on A is m-γ-tempered, and the smooth crossed product $G \rtimes^\gamma A$ is m-convex by Theorem 3.1.7.

We show that this action is *not* strongly m-γ-tempered. Let e_i be the element $(0, \ldots 0, 1, 0, \ldots) \in \mathbb{Q}^\infty$ where the 1 is in the ith spot. Let $\delta_{\vec{q}} \in A$ be the delta function at $\vec{q} \in \mathbb{Q}^\infty$. Let $q \in \mathbb{Q}$ with $q > 1$ and let $\vec{q_i}$ denote $(1, \ldots 1, q, 1, \ldots) \in G$, where the q is in the ith spot. Then

$$\alpha_{\vec{q_1}}(\delta_{e_1}) * \ldots \alpha_{\vec{q_n}}(\delta_{e_n}) = \delta_{(q,0,\ldots)} * \ldots \delta_{(0,\ldots,q,0,\ldots)} = \delta_{(q,\ldots q,0,\ldots)},$$

where n q's occur. The norm in A of $\delta_{(q,\ldots q,0,\ldots)}$ is precisely $\omega(q, \ldots q, 0, \ldots) = (1+q)^n$. If the action were strongly m-tempered, we would have this norm bounded by

$$\begin{aligned} &\max_{\tau_1 + \ldots \tau_n \le d} \gamma^{\tau_1}(\vec{q_1}) \ldots \gamma^{\tau_n}(\vec{q_n}) \| \delta_{e_1} \|_1 \ldots \| \delta_{e_n} \|_1 \\ &\le \quad \max \quad q^{\tau_1} \ldots q^{\tau_n} \omega(e_1) \ldots \omega(e_n) \le q^d 2^n \end{aligned}$$

times some fixed constant C to the nth power. This would have to hold for any q, which cannot be since $(1+q)^n$ is not bounded by $q^d 2^n C^n$ as q tends to infinity. Hence the action α of G on A is m-γ-tempered but not strongly m-γ-tempered.

Question 3.1.30. If G is a compactly generated group, do the notions of m-tempered and strongly m-tempered coincide ?

§3.2 Non m-convex Group Algebras

We construct several non m-convex group algebras, using sub-polynomial scales which are not m-sub-polynomial.

Let G be a locally compact compactly generated group, and let τ be the word gauge on G. Define scales σ_k on G by

$$\sigma_k(g) = e^{\tau(g)^k} \tag{3.2.1}$$

Proposition 3.2.2. *If G is compact, then every σ_k is equivalent to 1. Assume G is not compact. Then the scales σ_k form a family of increasing inequivalent sub-polynomial scales. If k is zero, $\sigma_k \equiv 1$, and if $k = 1$, σ_k is the exponentiated word gauge. If $k \geq 2$, then σ_k is not m-sub-polynomial.*

Proof. Assume G is not compact. Clearly $\sigma_{k+1} \geq \sigma_k$ since τ is integer valued. We show that σ_k cannot dominate σ_{k+1}. Assume that

$$\sigma_{k+1}(g) \leq C\sigma_k^d(g), \qquad g \in G.$$

Since G is not compact, τ can take on arbitrarily large integer values. So we have $e^{m^{k+1}} \leq Ce^{dm^k}$ for arbitrarily large values of m. This is impossible since C and d are fixed.

We show that if $k \geq 2$, then σ_k is not m-sub-polynomial. Assume that

$$\sigma_k(g_1 \ldots g_n) \leq C^n \sigma_k^d(g_1) \ldots \sigma_k^d(g_n). \tag{3.2.3}$$

For $n \in \mathbb{N}$, let $g_1, \ldots g_n$ be elements of the generating set U, such that $\tau(g_1 \ldots g_n) = n$. Then $\sigma_k(g_i) = e$ and by (3.2.3) we have

$$e^{n^k} \leq C^n e^{dn} \tag{3.2.4}$$

But since G is not compact, n can be taken arbitrarily large with d and C fixed. So (3.2.4) will be violated if $k \geq 2$. Hence σ_k is not m-sub-polynomial if $k \geq 2$.

We show that each σ_k is sub-polynomial. To do this, we show that

$$\tau^k(gh) \leq 2^k(\tau^k(g) + \tau^k(h)). \tag{3.2.5}$$

By the subadditivity of τ, it suffices to show that $(a+b)^k \leq 2^k(a^k+b^k)$ for nonnegative integers a and b. For this, it suffices to show that the real valued function $f\colon [0,\infty) \times [0,\infty) \to \mathbb{R}^+$ given by

$$f(x,y) = \frac{(x+y)^k}{x^k + y^k}$$

is bounded by 2^k. The function f is globally bounded since it is continuous and $\lim_{z\to\infty}(1+z)^k/(1+z^k)$ is one. Setting the first partial derivatives of f to zero, we see that f has a local maximum only if $x = \pm y$. If $x = y$, then $f(x,y) = 2^k$. So it follows that $f \leq 2^k$, and in fact that 2^k is the best bound. $\square$

Definition 3.2.6. If G is a compactly generated Lie group and $k \geq 1$, we let $\mathcal{S}^k(G)$ be the Schwartz algebra corresponding to the scale σ_k. We let $L^k(G)$ denote the Fréchet *-algebra $L_1^{\sigma_k}(G)$.

Note that σ_k dominates the exponentiated word weight σ_1, and hence dominates every weight on G - so in particular σ_k bounds Ad and $\mathcal{S}^k(G)$ is a Fréchet *-algebra.

Let G be any finitely generated discrete group which is not finite. For example, G could be the discrete subgroup

$$G = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix} \tag{3.2.7}$$

of the three dimensional Heisenberg Lie group. Let $k \geq 2$. Then $\mathcal{S}^k(G)$ in not m-convex by the following proposition and Proposition 3.2.2. Note that for the group (3.2.7) this gives an example of a noncommutative non m-convex Fréchet *-algebra.

Theorem 3.2.8. *Assume that G is a discrete group. Let σ be a sub-polynomial scale on G. Then the convolution algebra $\mathcal{S}^\sigma(G)$ is m-convex if and only if σ is m-sub-polynomial.*

Proof. By Theorem 3.1.7, it suffices to show that if $\mathcal{S}^\sigma(G)$ is m-convex, then σ is m-sub-polynomial. Let e_g be the function in $\mathcal{S}^\sigma(G)$ defined by

$$e_g(h) = \begin{cases} 1 & \text{if } h = g \\ 0 & \text{if } h \neq \end{cases} \tag{3.2.9}$$

For convenience, we replace σ with the equivalent scale $1+\sigma$ so that $\sigma \geq 1$. A family of increasing seminorms for $\mathcal{S}^\sigma(G)$ is given by

$$\| \varphi \|_m = \sum_{g \in G} \sigma^m(g) \, | \varphi(g) \, |.$$

Then

$$\begin{aligned} \left(\frac{\| e_{g_1} \cdots e_{g_n} \|_m}{\| e_{g_1} \|_k \cdots \| e_{g_n} \|_k} \right)^{1/n} &= \left(\frac{\| e_{g_1 \ldots g_n} \|_m}{\| e_{g_1} \|_k \cdots \| e_{g_n} \|_k} \right)^{1/n} \\ &= \left(\frac{\sigma^m(g_1 \ldots g_n)}{\sigma^k(g_1) \ldots \sigma^k(g_n)} \right)^{1/n}. \end{aligned} \tag{3.2.10}$$

By Theorem 3.1.4, the m-convexity of $\mathcal{S}^\sigma(G)$ implies that for all m there is some k such that (3.2.10) is bounded by a constant C. In particular, taking $m = 1$ we see that

$$\sigma(g_1 \ldots g_n) \leq C^n \sigma^k(g_1) \ldots \sigma^k(g_n) \tag{3.2.11}$$

and σ is m-sub-polynomial. $\square$

Question 3.2.12. Is it true for a general Lie group G that $\mathcal{S}^\sigma(G)$ is m-convex if and only if σ is m-sub-polynomial ?

We give some examples of non m-convex Schwartz algebras for connected Lie groups.

Proposition 3.2.13. *Let G be the Abelian group $\mathbb{R}^N$. Let σ_k be the non m-sub-polynomial scale defined above for any $k \geq 2$. Then σ_k is equivalent to the sub-polynomial scale*

$$\delta_k(x) = e^{|x|^k}$$

on $\mathbb{R}^N$. The convolution algebras $\mathcal{S}^k(G)$ and $L^k(G)$ are not m-convex.

Proof. To see that $\delta_k \sim \sigma_k$, take as generating set for G the open unit ball U. It suffices to show that $|x| \leq \tau(x) \leq |x| + 1$, where τ is the word gauge corresponding to U. If $\tau(x) = n$, then $x = x_1 + \ldots x_n$ with $x_i \in U$. Hence $|x| \leq n$ and we have $|x| \leq \tau(x)$ for all x. If $|x| = c$, let n be the smallest integer greater than or equal to c. Then $|x/n| < 1$ and $\tau(x) \leq n \leq c+1$.

To show that the Schwartz algebra is not m-convex, we use Theorem 3.1.4. To simplify notation, we denote δ_k by δ, where k is assumed greater than or equal to 2. We topologize $\mathcal{S}^k(G)$ by the increasing family of seminorms

$$\| \psi \|'_m = \max \int \delta^m(x) |X^\gamma \psi(x)| dx, \tag{3.2.14}$$

and topologize $L^k(G)$ by the increasing family $\| \quad \|_m$ of seminorms

$$\| \psi \|_m = \int_G \delta^m(x)\, |\psi(x)|\, dx. \tag{3.2.15}$$

Clearly $\| \psi \|_m \leq \| \psi \|'_m$ for $\psi \in \mathcal{S}^k(G)$.

Let ψ be any positive C^∞-function satisfying

$$\psi(x) = \begin{cases} 0 & |x| > 2 \\ 1 & |x| \leq 1 \end{cases} \tag{3.2.16}$$

Let u be a unit vector. Define $\psi_n(x) = \psi(x - nu)$. If $k \geq 2$, we show that for any $l \in \mathbb{N}$ and for any $C > 0$, there is some n sufficiently large such that

$$\underbrace{\left(\frac{\| \psi_1 \dots \psi_1 \|_1}{\| \psi_1 \|'_l \dots \| \psi_1 \|'_l} \right)}_{n\text{-times}}^{1/n} > C \tag{3.2.17}$$

By Theorem 3.1.4, this will establish that $\mathcal{S}^k(G)$ is not m-convex. And since the prime norms dominate the unprimed norms, it will also establish that $L^k(G)$ is not m-convex.

Since the denominator of (3.2.17) is just $\| \psi_1 \|'_l$, it suffices to show that $\| \psi_1^{*n} \|_1^{1/n}$ tends to infinity as $n \longrightarrow \infty$. We estimate $\| \psi_1^{*n} \|_1$.

$$\begin{aligned} \| \psi_1^{*n} \|_1 &= \int_G \delta(x) |\psi_1^{*n}(x)| dx \\ &= \int_G \delta(x + nu) |\psi^{*n}(x)| dx. \end{aligned} \tag{3.2.18}$$

We write

$$|\psi^{*n}(x)| = \int_G \cdots \int_G \psi(x_1) \dots \psi(x_{n-1}) \psi(x - x_1 - \dots x_{n-1}) dx_1 \dots dx_{n-1} \tag{3.2.19}$$

since ψ is positive. For definiteness, we choose $|x|$ to be the maximum of the components of the vector x. Let W be a neighborhood of 0 such that W^n is contained in the unit ball with respect to $| \quad |$, and the Lebesgue measure of W is greater than or equal to $1/n^N$ (For example W could be the unit ball of $\mathbb{R}^N$ to the $1/n$th power.) If, in the integral (3.2.19), we restrict $x_1, \dots x_{n-1}$ to be within the set W, then $\psi(x_i) = 1$ always. Thus

$$|\psi^{*n}(x)| \geq \int_{W^{n-1}} \psi(x - x_1 - \dots x_{n-1}) dx_1 \dots dx_{n-1}. \tag{3.2.20}$$

If we make the further restriction that $x \in W$, then the integrand of (3.2.20) is greater than or equal to one and

$$|\psi^{*n}(x)| \geq \int \qquad dx_1 \dots dx_{n-1} \geq 1/n^{(n-1)N}. \tag{3.2.21}$$

Plugging into (3.2.18), we see that

$$\begin{aligned} \| \psi_1^{*n} \|_1 &\geq \int_W \delta(x+nu)|\psi * \ldots \psi(x)|dx \\ &\geq \inf_{x\in W} \delta(x+nu) \int_W 1/n^{(n-1)N} dx \geq \delta((n-1)u)/n^{nN} = e^{(n-1)^k}/n^{nN}. \end{aligned} \tag{3.2.22}$$

Thus

$$\| \psi_1^{*n} \|_1^{1/n} \geq \frac{e^{(n-1)^{k-1}}}{(n)^N}. \tag{3.2.23}$$

Since $k \geq 2$, $e^{(n-1)^{k-1}}$ grows much faster than $(n)^N$. So given any $C > 0$, it is always possible to find an n sufficiently large so that (3.2.17) holds. This proves that $\mathcal{S}^k(G)$ and $L^k(G)$ are not m-convex by Theorem 3.1.4, and completes the proof of Proposition 3.2.13 $\square$

For other constructions of non m-convex Fréchet algebras, see [Ar], [RZ, §7], [Ze, Theorem 26], and [Pe].

§4 Conditions for a *-algebra

Let σ be a weight or a gauge on G that bounds Ad. The main result of this section is that if A_1 is a Fréchet *-algebra, with a tempered and strongly continuous action of G by *-automorphisms, and A is the set of C^∞-vectors for the action of G on A_1, then $G \rtimes^\sigma A$ is a Fréchet *-algebra (see Corollary 4.9 below). We first show that a tempered action of G on A_1 gives a tempered action of G on A. This fact will prove useful in examples, since the temperedness of an action on A_1 is often easier to verify than the temperedness on A. We also prove the appropriate m-versions of our results.

See the beginning of §1.3 for the definition of a Fréchet *-algebra. We remark that for a Fréchet *-algebra A, there always exists a set of *-isometric seminorms for A. Simply define $\| a \|_m' = \max(\| a \|_m, \| a^* \|_m)$. (**-isometric* means $\| a^* \|_m' = \| a \|_m'$ for all $a \in A$ and $m \in \mathbb{N}$.) Similarly, an m-convex Fréchet *-algebra is topologized by a family of submultiplicative *-isometric seminorms. We say that G acts by **-automorphisms* on A if $\alpha_g(a^*) = \alpha_g(a)^*$ for $a \in A$ and $g \in G$.

Let A be a Fréchet algebra with tempered and strongly continuous action of G. Even if A is a Fréchet *-algebra and G acts by *-automorphisms, the convolution algebra $G \rtimes A$ may not be closed under the involution $\varphi^*(g) = \Delta(g)\alpha_g(\varphi(g^{-1})^*)$. For example, let $G = \mathbb{R}$ with σ given by the absolute value, and let $A = C_c(\mathbb{R})$ where $\mathbb{R}$ acts by translation. Then

$G \rtimes A = \mathcal{S}(\mathbb{R}, C_0(\mathbb{R}))$, the Schwartz functions on $\mathbb{R}$ taking values in $C_0(\mathbb{R})$. The function

$$\psi(r,s) = e^{-r^2} \frac{1}{1+|s|} \tag{4.1}$$

is in $\mathcal{S}(\mathbb{R}, C_0(\mathbb{R}))$, but

$$\psi^*(r,s) = e^{-r^2} \frac{1}{1+|s-r|}$$

is not differentiable in the variable r. In this case, note that $G \rtimes A$ would be closed under the involution if the action of G on A were differentiable.

Let H be a Lie group, and assume $G \subseteq H$ with differentiable inclusion map. Let A be the set of C^∞-vectors for a strongly continuous action of H on a Fréchet algebra A_1. (We shall always assume that the action of G on A_1 is then given by the restriction of this action of H.) Let $X_1, ...X_p$ be a basis for the Lie algebra $\mathfrak{H}$ of H, where p is the dimension of $\mathfrak{H}$. If $a \in A$ and $\gamma \in \mathbb{N}^p$, let $X^\gamma a$ be given by formula (1.2.1) with $\beta = \alpha$ and $q = p$, where α denotes the action of H on A. The algebra A is topologized by the seminorms

$$\| a \|_{l,m} = \max_{|\gamma| \leq l} \| X^\gamma a \|_m . \tag{4.2}$$

In this topology, A is a Fréchet algebra (m-convex if A_1 is), and is a dense subalgebra of A_1 with continuous inclusion. Also, the Lie group H leaves A invariant and acts pointwise differentiably on A by continuous automorphisms. See Theorem A.2 in the appendix for proofs of these facts.

Definition 4.3. We say that a scale σ on G *bounds Ad on H* if there exists $C, D \geq 0$ and $d \in \mathbb{N}$ such that

$$\| Ad_g \|_{\mathfrak{H}} \leq C\sigma^d(g) + D, \tag{4.4}$$

for each $g \in G$. Here the norm is taken as an operator on the Lie algebra $\mathfrak{H}$ of H.

Example 4.5. We give an example of a scale σ on G which bounds Ad on G but not on H. Let G be the subgroup of integer valued matrices of the three dimensional Heisenberg Lie group

$$H = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} \\ 0 & 0 & 1 \end{pmatrix}.$$

If σ is the trivial scale, then σ of course bounds Ad on G since G is discrete. However, the norm of Ad_g as an operator on $\mathfrak{H}$ is equivalent to the sum of the two matrix entries just above the diagonal (see Example 1.5.15), so $\| Ad_g \|_{\mathfrak{H}}$ is unbounded for $g \in G$ and σ does not bound Ad on H.

We prove the following useful theorem. Recall that $\sigma_-(g) = \sigma(g^{-1})$.

Theorem 4.6. *Let A_1 be an (m-convex) Fréchet algebra with strongly continuous action of a Lie group H. Let G be a Lie group and assume $G \subseteq H$ with differentiable inclusion map. Let σ be a scale on G such that σ_- bounds Ad on H. Assume that A_1 is an (m-convex) Fréchet algebra on which the action of G is σ-tempered (m-σ-tempered). Then the algebra A of C^∞-vectors for the action of H on A_1 is an (m-convex) Fréchet algebra, and the action of G on A is σ-tempered (m-σ-tempered).*

*If A_1 is a Fréchet *-algebra and H acts by *-automorphisms on A_1, then A is a Fréchet *-algebra for which the above statements hold.*

Proof. For the first and last statements, see Theorem A.2. We prove the second statement. To simplify notation, assume that $\sigma \geq 1$, by replacing σ with $1+\sigma$ or $\max(\sigma, 1)$. Since σ_- bounds Ad on H, by Lemma 2.2.1 we have

$$\| X^\gamma \alpha_g(a) \|_m \leq C\sigma^d(g) \sum_{\beta \leq \gamma} \| \alpha_g(X^\beta(a)) \|_m, \quad a \in A, \quad g \in G. \tag{4.7}$$

Therefore

$$\| \alpha_g(a) \|_{l,m} \leq K\sigma^d(g) \max_{|\beta| \leq l} \| \alpha_g(X^\beta(a)) \|_m \, . \tag{4.8}$$

If the action of G on A_1 is tempered, (4.8) tells us that $\| \alpha_g(a) \|_{l,m} \leq \tilde{K}\sigma^{d+k}(g) \| a \|_{l,t}$ for some $t \geq m$, so the action of G on A is tempered. If the action of G on A_1 is m-tempered, and $\| \ \|_m$ are (G, σ)-stable submultiplicative seminorms for A_1, then we have $\| \alpha_g(a) \|_{l,m} \leq \tilde{K}\sigma^{d+k}(g) \| a \|_{l,m}$, so the seminorms $\| \ \|_{l,m}$ on A are (G,σ)-stable. Also, a constant times $\| \ \|_{l,m}$ gives (G,σ)-stable submultiplicative seminorms for A (see (A.6) and following remarks), so the action of G on A is m-tempered. □

Corollary 4.9. *Let G, H, and A_1 be as in Theorem 4.6 above, and assume that A_1 is an (m-convex) Fréchet *-algebra on which H acts by *-automorphisms. Let A be the (m-convex) Fréchet *-algebra of C^∞-vectors for the action of H on A_1. If the action of G on A_1 is σ-tempered (m-σ-tempered), and σ is a sub-polynomial (m-sub-polynomial) scale on G which bounds Ad on H, and satisfies $\sigma \sim \sigma_-$ (for example, σ could be a weight or a gauge on G which bounds Ad on H), then the convolution algebra $G \rtimes^\sigma A$ is an (m-convex) Fréchet *-algebra.*

Proof. For everything except the *-algebra part, see Theorems 2.2.6, 3.1.7, and 4.6 above. It remains to show that involution is well defined and continuous on the algebra $G \rtimes A$. We

let $\| \ \|_d$ be seminorms (4.2) for A, and let $\| \ \|_{m,\gamma,d}$ be seminorms (2.1.1) for $G \rtimes A$ as in Definition 2.1.0. Just as in the proof of Theorem 4.6 above, we may assume that $\sigma \geq 1$. Recall that $\psi^*(g) = \alpha_g(\psi(g^{-1})^*)\Delta(g)$, where Δ is the modular function for G. We must show that

$$\| \psi^* \|_{m,\gamma,d} = \int_G \| \sigma^m X^\gamma \psi^*(g) \|_d \, dg \tag{4.10}$$

is bounded by a linear combination of seminorms in ψ. We do most of the work in the following lemma.

Lemma 4.11. *Let $\gamma \in \mathbb{N}^q$ (where q is the dimension of the Lie algebra $\mathfrak{G}$ of G) and assume that $\sigma \geq 1$ bounds Ad on G, and $\sigma \sim \sigma_-$. Then there is some $j \in \mathbb{N}$ so that*

$$\| (X^\gamma \psi^*)(g) \|_m \leq \sum_{\beta, \tilde{\beta} \leq \gamma} C\sigma^j(g) \| \alpha_g(X^\beta((X^{\tilde{\beta}}\psi)(g^{-1})))^* \|_m, \tag{4.12}$$

where the operator $X^{\tilde{\beta}}$ is from the Lie algebra $\mathfrak{G}$ of G acting by left translation on $\psi \in G \rtimes A$, and X^β is from $\mathfrak{G}$ acting via α and the inclusion $\mathfrak{G} \subseteq \mathfrak{H}$.

Proof. It is well known fact that the modular function $\Delta: G \longrightarrow \mathbb{R}^+$ is differentiable [War, Vol. I]. We show that all of the derivatives of Δ are bounded by some σ^p. Let $\tilde{K}$ be a constant such that $|(X^\beta \Delta)(e)| \leq \tilde{K}$ for all $\beta \leq \gamma$. Since Δ is the absolute value of the determinant of Ad [War, Vol. I, Appendix] and σ bounds Ad, there are C and p such that

$$\Delta(g) \leq C\sigma^p(g). \tag{4.13}$$

So by the multiplicativity of Δ, we have

$$\begin{aligned} |(X^\beta \Delta)(g)| &\leq \tilde{K}\Delta(g) \\ &\leq K\sigma^p(g) \quad \text{by (4.13)} \end{aligned} \tag{4.14}$$

where $K = \tilde{K}C$ and $\beta \leq \gamma$. This inequality will be very useful in our calculations.

Using the product rule and the chain rule, and the formula $\psi^*(g) = \alpha_g(\psi(g^{-1})^*)\Delta(g)$, we see that for $\gamma \in \mathbb{N}^q$,

$$\begin{aligned} &(X^\gamma \psi^*)(g) \\ &= \sum \alpha_g(X^{\beta_1}((X^{\beta_2}\psi)(g^{-1})^*))(X^{\beta_3}\Delta)(g))P_{\beta_1,\beta_2,\beta_3}((Ad_{g^{-1}})_{ji}) \end{aligned} \tag{4.15}$$

where the $P_{\beta_1,\beta_2,\beta_3}$ are polynomials with degree bounded by $|\gamma|$. Note that the matrix entries $(Ad_{g^{-1}})_{ij}$ in (4.15) are from $Ad_{g^{-1}}$ as an operator on the Lie algebra of G, not just H (since X^γ and all the other differential operators in (4.15) come from $\mathfrak{G}$). Since $\sigma_- \sim \sigma$, and σ bounds Ad on G, there is some $s \in \mathbb{N}$ and C_1 such that

$$|P_{\beta_1,\beta_2,\beta_3}((Ad_{g^{-1}})_{ji})| \le C_1 \sigma^s(g). \tag{4.16}$$

Estimating $\| X^\gamma \psi^*(g) \|_m$ using (4.15), (4.16), and (4.14), we get

$$\| (X^\gamma \psi^*)(g) \|_m \le \sum_{\beta_1,\beta_2 \le \gamma} C_1 K \sigma^{s+p}(g) \| \alpha_g(X^{\beta_1}((X^{\beta_2}\psi)(g^{-1})))^* \|_m \tag{4.17}$$

Taking $j = p + s$ and $C = C_1 K$ yields (4.12) and proves Lemma 4.11. $\square$

We have

$$\begin{aligned}
&\| \Delta(X^\gamma \psi^*)(g^{-1}) \|_m \\
&\le \sum_{\beta,\tilde\beta \le \gamma} C_1 \sigma^j(g) \| \alpha_{g^{-1}}(X^\beta((X^{\tilde\beta}\psi)(g)))^* \|_m \qquad \text{by Lemma 4.11, (4.14)} \\
&\le \sum_{\beta,\tilde\beta \le \gamma} C_2 \sigma^r(g) \| X^\beta((X^{\tilde\beta}\psi)(g))^* \|_m \quad \text{tempered action, } \sigma_- \sim \sigma \\
&\le \sum_{\tilde\beta \le \gamma} C_3 \| \sigma^r (X^{\tilde\beta}\psi)(g) \|_p \quad \text{continuity of } * \text{ and of differentiation on } A,
\end{aligned} \tag{4.18}$$

for constants $C_1, C_2, C_3 > 0$, and sufficiently large integers r, p. So we have

$$\begin{aligned}
\| \psi^* \|_{m,\gamma,d} &= \int_G \| \sigma^m (X^\gamma \psi^*)(g) \|_d \, dg \\
&= \int_G \| \sigma^m \Delta(X^\gamma \psi^*)(g^{-1}) \|_d \, dg \\
&\le \int_G \Big(\sum_{\tilde\beta \le \gamma} C_3 \| \sigma^{m+r}(X^{\tilde\beta}\psi)(g) \|_p \Big) dg \ (4.18) \\
&= \sum_{\tilde\beta \le \gamma} C_3 \| \psi \|_{m+r,\tilde\beta,p} \, .
\end{aligned} \tag{4.19}$$

Thus $*$ is continuous. This proves Corollary 4.9. $\square$

§5 Examples and Scaled G-Spaces

We develop the notion of a scaled (G, ω)-space, where ω is a sub-polynomial scale on G, which will give us many examples of dense subalgebras (with tempered and m-tempered

functions on M, vanishing at infinity with sup norm.) We then look at examples of scaled (G, ω)-spaces, and show that we can form the smooth crossed product $G \rtimes \mathcal{S}(M)$. We also make note of some examples of dense subalgebras of B in the case that B is a noncommutative C*-algebra.

To simplify our formulas, unless specified otherwise, we assume that $\sigma \geq 1$. This can always be achieved by replacing σ with one of the equivalent scales $\max(\sigma, 1)$ or $1 + \sigma$.

Definition 5.1. Let M be a locally compact space. Let $\sigma \geq 1$ be a scale on M which is bounded on compact subsets of M.

Let $C^\sigma(M)$ be the set of functions

$$\{ f \in C_0(M) \mid \| \sigma^d f \|_\infty < \infty \quad \forall d \in \mathbb{N} \},$$

which we call the continuous functions which vanish σ-rapidly. It follows that $\sigma^d f$ vanishes at infinity for every $d \in \mathbb{N}$, even if σ is not a proper map. We topologize $C^\sigma(M)$ by the seminorms

$$\| f \|_d = \| \sigma^d f \|_\infty, \qquad d = 0, 1, \ldots \tag{5.2}$$

We will usually denote the seminorm $\| \quad \|_0$ by $\| \quad \|_\infty$. If σ and τ are equivalent scales, then $C^\sigma(M) = C^\tau(M)$. If $\{f_n\}$ is a Cauchy sequence in $C^\sigma(M)$, then $f_n \longrightarrow f_0$ in $\| \quad \|_\infty$ for some $f_0 \in C_0(M)$. Note

$$\sigma^d(m)|f_0(m)| \leq \sigma^d(m)|f_0(m) - f_k(m)| + \| f_k - f_n \|_d + \| f_n \|_d \tag{*}$$

for $m \in M$, $k, n \in \mathbb{N}$. Let N be so large that

$$n, k \geq N \Longrightarrow \|f_k - f_n\|_d < 1.$$

Take $n = N$ in $(*)$ and fix $m \in M$. By letting k run in $(*)$, we see that $\sigma^d(m)|f_0(m)| \leq 2 + \| f_N \|_d$. Hence $f_0 \in C^\sigma(M)$. Similar arguments show that $f_n \longrightarrow f_0$ in $C^\sigma(M)$, so $C^\sigma(M)$ is complete. The space $C^\sigma(M)$ is an m-convex (see §3) Fréchet *-algebra, since

$$\begin{aligned} \| f_1 \ldots f_n \|_d &= \| \sigma^d f_1 \ldots f_n \|_\infty \\ &\leq \| f_1 \|_d \| f_2 \|_\infty \cdots \| f_n \|_\infty . \end{aligned} \tag{5.3}$$

Since σ is bounded on compact sets, the compactly supported continuous functions $C_c(M)$ are contained in $C^\sigma(M)$, so $C^\sigma(M)$ is dense in $C_0(M)$. The seminorms (5.2) are continuous for the inductive limit topology on $C_c(M)$.

Remark 5.4. If σ is not bounded on compact subsets, then $C^\sigma(M)$ may not be dense in $C_0(M)$. For let $M = \mathbb{R}$ and set

$$\sigma(r) = \begin{cases} \frac{1}{|r|} + |r| & r \neq 0. \\ 0 & r = 0. \end{cases} \tag{5.5}$$

Then every element of $C^\sigma(M)$ vanishes at 0.

Definition 5.6. Now assume that M is an H-set, with H a Lie group. If $f \in C^\sigma(M)$, define $\alpha_h(f)(m) = f(h^{-1}m)$. We impose a condition that makes α a strongly continuous action of H on $C^\sigma(M)$. We say that σ is *uniformly H-translationally equivalent* if for every compact subset $K \subseteq H$, there exists $l \in \mathbb{N}$ and $C > 0$ so that

$$\sigma(hm) \leq C\sigma^l(m), \qquad h \in K, m \in M. \tag{5.7}$$

Just as we saw in Theorem 1.2.11 that for scales on a group, translational equivalence implies uniform translational equivalence, it is probably also true that the notion of H-translational equivalence (which we haven't stated) implies uniform H-translational equivalence. We shall not be needing this, however. We show that if σ is uniformly H-translationally equivalent, then H acts on $C^\sigma(M)$ by continuous automorphisms, and moreover that the action is strongly continuous. For $h \in K$, $f \in C^\sigma(M)$, we have

$$\begin{aligned} \| \alpha_h(f) \|_d &= \| \sigma^d \alpha_h(f) \|_\infty = \sup_{m \in M} \sigma^d(m)|f(h^{-1}m)| \\ &= \sup_{m \in M} \sigma^d(hm)|f(m)| \leq C^d \| \sigma^{ld} f \|_\infty = C^d \| f \|_{ld} \, . \end{aligned} \tag{5.8}$$

So H clearly leaves $C^\sigma(M)$ invariant and acts by bounded, and hence continuous, automorphisms on $C^\sigma(M)$. Since the action of H on $C_c(M)$ is strongly continuous for the inductive limit topology, a standard argument shows that the action of H on $C^\sigma(M)$ is strongly continuous.

If σ is not uniformly H-translationally equivalent, then H may not leave $C^\sigma(M)$ invariant as the following example shows.

Example 5.9. Let $M = \mathbb{R}^+$, $\sigma(r) = 1+|r|$, and $H = \mathbb{Z}/2\mathbb{Z}$. Let H act on M via $\alpha(r) = 1/r$. Then σ is not uniformly H-translationally equivalent, since (5.7) is not satisfied. The function $f(r) = \min(r, e^{1-r^2})$ is in $C^\sigma(M)$, but $\alpha(f)$ is not. This is because e^{1-r^2} vanishes faster at infinity than any power of $1/r$, but $1/r$ does not.

If σ is uniformly H-translationally equivalent, we define the *σ-rapidly vanishing H-Schwartz functions on* (M, σ), denoted by $\mathcal{S}^\sigma(M)$, to be the set of C^∞ vectors for the action

of H on $C^\sigma(M)$. Since what group we are using is usually clear, we often write $\mathcal{S}^\sigma(M)$ to abbreviate $\mathcal{S}^\sigma_H(M)$. If it is clear both what group and scale we are using, we will often simply write $\mathcal{S}(M)$. If M happens to be the group H, we may write $\mathcal{S}^\sigma_\infty(H)$, with subscript ∞ to denote the sup norm, to contrast the space with the L^1 Schwartz functions $\mathcal{S}^\sigma_1(H)$ on H.

The space $\mathcal{S}^\sigma(M)$ has a natural Fréchet topology, and is dense in $C^\sigma(M)$ and hence dense in $C_0(M)$. Also, the action of H by left translation leaves $\mathcal{S}^\sigma(M)$ invariant and the action of H on $\mathcal{S}^\sigma(M)$ by continuous automorphisms is pointwise differentiable. The Fréchet space $\mathcal{S}^\sigma(M)$ is also an m-convex Fréchet *-algebra. See Theorem A.2 below for proofs of these facts.

Let p be the dimension of the Lie algebra of H. The topology on $\mathcal{S}^\sigma(M)$ is given by the norms

$$\| f \|_{l,\gamma} = \| \sigma^l X^\gamma f \|_\infty, \tag{5.10}$$

where $\gamma \in \mathbb{N}^p$, and the notation $X^\gamma f$ is defined in equation (1.2.1) with action $\beta = \alpha$ and $q = p$.

Example 5.11. Let $M = H = \mathbb{R}$, where H acts by translation, and let $\sigma(r) = 1 + |r|$. Then σ is uniformly H-translationally equivalent and $\mathcal{S}^\sigma(M)$ is just the standard set of Schwartz functions on $\mathbb{R}$.

We would like to define a smooth crossed product $G \rtimes \mathcal{S}^\sigma_H(M)$ where G is an arbitrary subgroup of H. For example, we would like to be able to define smooth crossed products $\mathbb{R} \rtimes \mathcal{S}(\mathbb{R})$ and $\mathbb{Z} \rtimes \mathcal{S}(\mathbb{R})$. To facilitate this, we make the following definition.

Definition 5.12. Let G be a Lie group, which is a subgroup of H with differentiable inclusion map. Let σ be a uniformly H-translationally equivalent scale on M. Then we know that the action α of H on $\mathcal{S}^\sigma_H(M)$ by left translation is differentiable. This action clearly restricts to a differentiable action of G. Let $\omega \geq 1$ be a sub-polynomial scale on G. We impose a condition that makes α an m-ω-tempered action of G on $\mathcal{S}^\sigma_H(M)$. We say that (M, σ, H) *is a scaled* (G,ω)*-space* if ω_- bounds Ad on H (see Definition 4.3) (for example, ω could be a weight or a gauge on G that bounds Ad on H) and there exists $l \in \mathbb{N}$ and $C > 0$ so that

$$\sigma(gm) \leq C\omega^l(g)\sigma^l(m), \qquad g \in G, m \in M. \tag{5.13}$$

If we say that (M,σ) or simply M is a scaled (G,ω)-space, with no mention of a group H, then it will be implied that we are taking $H = G$. It is useful to note that condition (5.13) implies that σ is uniformly G-translationally equivalent, since ω is bounded on compact sets.

Example 5.14. For example, let $M = H = \mathbb{R}$, $\sigma(r) = 1 + |r|$, and $\mathcal{S}^\sigma_H(\mathbb{R}) = \mathcal{S}(\mathbb{R})$ as in Example 5.11. Let G be the closed subgroup $\mathbb{Z}$ of H, with $\omega(n) = e^{|n|}$ or $\omega(n) = 1 + |n|$. Then for either choice of ω, we have

$$\sigma(n + r) \leq 1 + |n| + |r| \leq \omega(n)\sigma(r).$$

So (M, σ, H) is a scaled (G, ω)-space. However, if we choose the weight $\omega \equiv 1$, then (M, σ, H) is *not* a scaled (G, ω)-space.

Let (M, σ, H) be a scaled (G, ω)-space. We show that the action of G on $\mathcal{S}^\sigma_H(M)$ is m-ω-tempered. For this, by Theorem 4.6 it suffices to show that the action of G on $C^\sigma(M)$ is m-ω-tempered. By the estimate (5.8) with C replaced by $C\omega^l(g)$, we have

$$\| \alpha_g(f) \|_d \leq C\omega^l(g) \| f \|_d \, .$$

Since each seminorm $\| \quad \|_d$ is submultiplicative (recall we are assuming $\sigma \geq 1$), this shows that the action of G on $C^\sigma(M)$ is m-ω-tempered.

We may form the convolution algebra $G \rtimes^\omega \mathcal{S}^\sigma_H(M)$, which is topologized by the seminorms

$$\| \varphi \|_{d,\gamma,l,\beta} = \int_G \sup_{m \in M} \Big(\omega^d(g)\sigma^l(m) |X^\gamma \tilde{X}^\beta \varphi(g, m)| \Big) dg, \tag{5.15}$$

where X^γ acts on the first argument of F, and $\tilde{X}^\beta$ on the second. We place the usual operations of multiplication and involution on $G \rtimes^\omega \mathcal{S}^\sigma_H(M)$:

$$\begin{aligned} \varphi * \psi(g, m) &= \int_G \varphi(h, m)\psi(h^{-1}g, h^{-1}m) dh \\ \varphi^*(g, m) &= \overline{\varphi}(g^{-1}, g^{-1}m)\Delta(g). \end{aligned} \tag{5.16}$$

Theorem 5.17. *Let (M, σ, H) be a scaled (G, ω)-space. Then the space $G \rtimes^\omega \mathcal{S}^\sigma_H(M)$ is a dense Fréchet subalgebra of $L^1(G, C_0(M))$, and of the C^*-crossed products $G \rtimes C_0(M)$ and $G \rtimes_r C_0(M)$. It is m-convex if ω is m-sub-polynomial, and is a *-algebra if $\omega \sim \omega_-$. (These last two conditions are satisfied if ω is a weight or gauge on G.)*

Proof. We have seen that the action of G on $\mathcal{S}^\sigma_H(M)$ is m-ω-tempered and differentiable. Hence by Theorem 2.2.6, we have the first statement of the theorem. Since $G \rtimes^\omega \mathcal{S}^\sigma_H(M)$ contains $C_c^\infty(G, \mathcal{S}^\sigma_H(M))$, we have the density statement. For the m-convexity, see Theorem 3.1.7. See Corollary 4.9 for the last statement. $\square$

Now we give several examples of scaled (G, ω)-spaces.

Example 5.18. In Example 5.14 with $\omega(n) = 1+|n|$, the resulting crossed product $G \rtimes \mathcal{S}(M)$ is $\mathbb{Z} \rtimes \mathcal{S}(\mathbb{R}) = \mathcal{S}(\mathbb{Z}, \mathcal{S}(\mathbb{R}))$, the standard set of Schwartz functions from $\mathbb{Z}$ to $\mathcal{S}(\mathbb{R})$.

Example 5.19. We note that the scale and weights we choose may depend on the action of G on M. For example, let $G = H = \mathbb{R}$, $M = \mathbb{R}$, and let G act by translation. Then the weight $\omega(r) = 1 + |r|$ and scale $\sigma(m) = 1 + |m|$ make (M, σ) a scaled (G, ω)-space. However, if the action of G on M is given by $r(m) = e^r m$, they do not. If we define a weight δ on G by $\delta(r) = e^{|r|}$, then (M, σ) is a scaled (G, δ)-space for this new action.

Example 5.20. Let $G = H$ be the $ax + b$ group, and let $M = \mathbb{R}$. Let G act on M by

$$\begin{pmatrix} e^{g_1} & g_2 \\ 0 & 1 \end{pmatrix} m = e^{g_1} m + g_2 \tag{5.21}$$

Let ω be the weight $\omega(g) = e^{|g_1|} + |e^{-g_1} g_2| + |g_2| + 1$ on G of Example 1.6.1. Let $\sigma(r) = 1 + |r|$ be the scale on $\mathbb{R}$. Then

$$\begin{aligned} \sigma(gm) &= (1 + |e^{g_1} m + g_2|) \leq (1 + \omega(g)|m| + \omega(g)) \\ &\leq 2\omega(g)(1 + |m|) = 2\omega(g)\sigma(m), \end{aligned} \tag{5.22}$$

so (M, σ) is a scaled (G, ω)-space. The resulting crossed product $G \rtimes \mathcal{S}(\mathbb{R})$ is isomorphic as a Fréchet space to $\mathcal{S}^\omega(G) \widehat{\otimes}_\pi \mathcal{S}(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ denotes the standard space of Schwartz functions on $\mathbb{R}$, $\mathcal{S}^\omega(G)$ is the group convolution algebra of Example 1.6.1, and $\widehat{\otimes}_\pi$ denotes the completed projective tensor product.

Example 5.23. We let $G = H$ be any closed subgroup of $GL(n, \mathbb{R})$, and let $\omega(A) = \max(\| A \|, \| A^{-1} \|)$ be the weight on G from Example 1.6.10. We consider two G-spaces.

(1) The group G acts on $\mathbb{R}^n$ by left multiplication.
(2) The group G acts on $M(n, \mathbb{R})$ by conjugation.

We define a scale σ on $M(n, \mathbb{R})$ by setting $\sigma(S) = 1 + \| S \|$, where $\| S \|$ denotes the operator norm of the matrix S. We define σ on $\mathbb{R}^n$ in a similar way. In each case (M, σ) is a scaled (G, ω)-space. We verify this for $M = M(n, \mathbb{R})$. Let $A \in G$ and $S \in M$. Then

$$\begin{aligned} \sigma(ASA^{-1}) =& 1 + \| ASA^{-1} \| \\ & \leq 1 + \| A \| \| S \| \| A^{-1} \| \leq \| A \| \| A^{-1} \| \, \sigma(S). \end{aligned}$$

Since $\omega^2(A) \geq \| A \| \| A^{-1} \|$, this shows that (M, σ) is a scaled (G, ω)-space.

Example 5.24. Let G be a Lie group with weight ω that bounds Ad on G. Let N be a subgroup of G and let (M, σ) be a scaled (N, ω)-space. Define an equivalence relation on $G \times M$ by $(g, m) \sim (\tilde{g}, \tilde{m})$ if and only if there is some $n \in N$ such that $\tilde{g} = gn^{-1}$ and $\tilde{m} = nm$. Let $M_G = G \times_N M$ denote the quotient space. Then G acts as a transformation group on M_G. The G-space M_G is *induced* from the N-space M.

We define a scale σ_G on M_G by

$$\sigma_G([g, m]) = \inf_{n \in N} \omega(gn^{-1})\sigma(nm). \tag{5.25}$$

Then $\sigma_G(\tilde{g}[(g, m)]) \leq \omega(\tilde{g})\sigma_G([(g, m)])$ so (M_G, σ_G) is a scaled (G, ω)-space. We let $\mathcal{S}(M_G)$ denote the corresponding G-Schwartz functions.

For a specific example, let $G = \mathbb{R}$, $N = \mathbb{Z}$ and let M be the circle $\mathbb{T}$. Let $\mathbb{Z}$ act on $\mathbb{T}$ by an irrational rotation θ. Let $\omega(r) = 1 + |r|$ be the weight on $\mathbb{R}$ and $\sigma \equiv 1$ be the constant scale on $\mathbb{T}$. Then $(M_\mathbb{R} = \mathbb{R} \times_\mathbb{Z} \mathbb{T}, \sigma_\mathbb{R})$ is a scaled $(\mathbb{R}, \omega)$-space. We may form $\mathcal{S}(M_\mathbb{R})$. Note that for $[(r, t)] \in M_\mathbb{R}$ we have $\sigma_\mathbb{R}([(r, t)]) = \inf_{n \in \mathbb{Z}} \omega(r - n)\sigma(n\theta + t) = \inf_{n \in \mathbb{Z}} \omega(r - n) \leq 2$.

Example 5.26. The scale $\sigma_0 \equiv 1$ on M always makes (M, σ_0) a scaled (G, ω)-space, for any group G and weight ω which bounds Ad on G. If M is a scaled H-space, with H properly containing G, then (M, σ_0, H) is a scaled (G, ω) space as long as ω bounds Ad on H.

Example 5.27. To see some cases where A is noncommutative, we could in general take $A = B$, or $A = B^\infty$, where B^∞ denotes the set of C^∞-vectors for the action of a Lie group G on a C*-algebra B. Let ω be any weight on G which bounds Ad. If $A = B$,

$$\| \alpha_g(a) \| = \| a \|, \tag{5.28}$$

so the action of G is always tempered and m-tempered. We saw in Theorem 4.6 that if $A = B^\infty$, then the action of G on A is still tempered and m-tempered. So in either case, we may form the m-convex Fréchet *-algebra $G \rtimes^\omega A$. We say that G is an elementary Abelian group if it is of the form $\mathbb{T}^n \times \mathbb{R}^m \times \mathbb{Z}^p \times F$, where F is a finite Abelian group. If G is an elementary Abelian group, $A = B^\infty$, and ω is any weight equivalent to the word gauge on G, then the algebra $G \rtimes^\omega A$ is precisely the dense subalgebra defined in [Bo, §2.1.4] with $E = A$. (We shall see in Theorem 6.8 and Proposition 6.13(2) below that although Bost uses the sup norm and I use the L^1 norm, we still get the same algebra.)

For a specific example where B is noncommutative, we let B be the two dimensional irrational rotation algebra A_θ. Here we think of A_θ as the C*-algebra generated by two

unitaries U, V with commutation relations

$$UV = e^{-2\pi i\theta} VU$$

We let $G = SL(2,\mathbb{Z})$ and ω be any weight on G. Let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}).$$

Then G acts on A_θ by

$$\alpha_g(U^n V^m) = \xi(g,n,m) U^{an+bm} V^{cn+dm}$$

where ξ is a map to $\mathbb{T}$ given by

$$\xi(g,n,m) = e^{(nm-(an+bm)(cn+dm))i\pi\theta/2}$$

We thus may form the dense m-convex Fréchet *-subalgebra $SL(2,\mathbb{Z}) \rtimes^\omega A_\theta^\infty$ $(= \mathcal{S}_1^\omega(SL(2,\mathbb{Z}), A_\theta))$ of the C*-algebra $SL(2,\mathbb{Z}) \rtimes A_\theta$.

§6 Sup Norm and L^r norm in Place of L^1 norm, and nuclearity

We define the L^r Schwartz functions on G for $1 \leq r \leq \infty$. We denote these by $\mathcal{S}_r^\sigma(G)$. We show that $\mathcal{S}_r^\sigma(G)$ is always contained in $\mathcal{S}_\infty^\sigma(G)$ with continuous inclusion map. If the reciprocal of the scale is in $L^p(G)$ for some $p \geq 1$, we show that $\mathcal{S}_1^\sigma(G) = \mathcal{S}_r^\sigma(G) = \mathcal{S}_\infty^\sigma(G)$. We show that in many of our examples, this condition is satisfied. We compare our algebras with those in [Jo], [Ji], and with Rader's algebra [War, Prop 8.3.7.14] and the zero Schwartz space for $SL(2,\mathbb{R})$ [Bar, §19]. Finally we prove that if the above integrability condition on $1/\sigma$ is satisfied, then $\mathcal{S}_r^\sigma(G)$ is a nuclear Fréchet space.

Definition 6.1. We defined $\mathcal{S}_\infty^\sigma(G)$ in §5. Let $1 \leq r < \infty$. Let $\sigma \geq 1$ be a scale on a Lie group G. Let $\mathcal{S}_r^\sigma(G)$ be the set of differentiable functions $\psi: G \to \mathbb{C}$ satisfying

$$(6.2) \qquad \| \psi \|_{m,\gamma}^{(r)} = \| \sigma^m X^\gamma \psi \|_r = \left(\int_G |\sigma^m X^\gamma \psi(g)|^r dg \right)^{1/r} < \infty$$

for each $\gamma \in \mathbb{N}^q$ and $m \in \mathbb{N}$. We call $\mathcal{S}_r^\sigma(G)$ the *σ-rapidly vanishing (L^r) Schwartz functions on G*. We topologize $\mathcal{S}_r^\sigma(G)$ by the seminorms (6.2). It is easily checked that if $\sigma_1 \sim \sigma_2$, then $\mathcal{S}_r^{\sigma_1}(G)$ is isomorphic to $\mathcal{S}_r^{\sigma_2}(G)$.

Definition 6.3. Let $\sigma \geq 1$ be any scale, and let $r \geq 1$. Define the *σ-rapidly vanishing L^r functions $L_r^\sigma(G)$ on G* to be the space of Borel measurable functions $f\colon G \to \mathbb{R}$ such that

$$\| f \|_d^{(r)} = \left(\int_G | \sigma^d f(g) |^r \, dg \right)^{1/r} \tag{6.4}$$

is finite for each $d \in \mathbb{N}$. Then $L_r^\sigma(G)$ is complete for the topology given by the seminorms $\| \;\; \|_d^{(r)}$ [Schw, §5]. Just as in §1 for $r = 1$, we have the following theorem.

Theorem 6.5. *Let σ be a translationally equivalent scale (for example a weight or a gauge) on a Lie group G. Then the action $\alpha_g(\varphi)(h) = \varphi(g^{-1}h)$ of G on $\mathcal{S}_r^\sigma(G)$ and $L_r^\sigma(G)$ by left translation is well defined, strongly continuous, and differentiable on $\mathcal{S}_r^\sigma(G)$. The Fréchet space of C^∞-vectors $L_r^\sigma(G)^\infty$ is naturally isomorphic to $\mathcal{S}_r^\sigma(G)$. Hence $\mathcal{S}_r^\sigma(G)$ is complete and a Fréchet space.*

Proof. The proof is the same as for the case $r = 1$ - see Theorems 1.2.21 and 2.1.5. $\square$

The Schwartz functions $\mathcal{S}_r^\sigma(G)$ may not be algebras if $1 < r \leq \infty$, even if σ is a weight or a gauge. For example, let $\sigma \equiv 1$. Then $\mathcal{S}_1^\sigma(G)$ is an algebra. We have $\mathcal{S}_\infty^\sigma(G) = C_0^\infty(G)$, where $C_0^\infty(G)$ is the set of differentiable functions ψ such that ψ and all its derivatives vanish at infinity. In this case, $\mathcal{S}_\infty^\sigma(G)$ is rarely an algebra for convolution. For example, take $G = \mathbb{Z}$. Let $\psi \in c_0(\mathbb{Z})$ be defined by

$$\psi(n) = \frac{1}{(1 + |n|)^{1/2}}. \tag{6.6}$$

Then

$$\begin{aligned} \psi * \psi(0) &= \sum_m \psi(m)\psi(0 - m) \\ &= \sum_m \frac{1}{(1 + |m|)^{1/2}} \frac{1}{(1 + |-m|)^{1/2}} \\ &= \sum_m \frac{1}{1 + |m|}, \end{aligned} \tag{6.7}$$

which diverges. (Note that in this case, we do not have $\mathcal{S}_\infty^\sigma(G) \subseteq C^*(G)$ or $\subseteq L^1(G)$.)

For an example when $\mathcal{S}_r^\sigma(G)$ is not an algebra for $r < \infty$, let $r = 2$ and $G = \mathbb{Z}$. Then $\mathcal{S}_2^\sigma(G) = l^2(\mathbb{Z})$ which is not a convolution algebra. (To see this, note that $l^2(\mathbb{Z}) \cong L^2(\mathbb{T})$ via the Fourier transform, which is not closed under pointwise multiplication.)

Theorem 6.8. *Let $\sigma \geq 1$ be a translationally equivalent scale on a Lie Group G. Then $\mathcal{S}_r^\sigma(G)$ is contained in $\mathcal{S}_\infty^\sigma(G)$ for any $r \geq 1$, with continuous inclusion map*

Assume in addition that there is some $p > 0$ such that

$$\int_G \frac{1}{\sigma^p(g)} dg < \infty. \tag{6.9}$$

Then $\mathcal{S}^\sigma_r(G) = \mathcal{S}^\sigma_1(G)$ as Fréchet spaces for all $r \in [1, \infty]$.

If σ satisfies condition (6.9), we say that $1/\sigma$ *is in some L^p-space.* Note that condition (1.5.2) is equivalent to (6.9).

Proof. By Theorem 6.5 and [DM, Thm 3.3], we may write any element of $\mathcal{S}^\sigma_r(G)$ as a finite sum of functions of the form

$$f * \psi(g) = \int f(h)\psi(h^{-1}g)dh,$$

where $f \in C^\infty_c(G)$ and $\psi \in \mathcal{S}^\sigma_r(G)$. By the chain rule and uniform translational equivalence of the scale (see Theorem 1.2.11),

$$\begin{aligned} |\sigma^m X^\gamma (f * \psi)(g)| &\leq \sum_{\beta \leq \gamma} \int |\, \sigma^m(g) p_\beta((Ad_{h^{-1}})_{ij}) f(h) X^\beta \psi(h^{-1}g)\,|\, dh \quad (2.2.3) \\ &\leq \sum_{\beta \leq \gamma} C \int |\, p_\beta((Ad_{h^{-1}})_{ij}) f(h) \sigma^{mk}(h^{-1}g) X^\beta \psi(h^{-1}g)\,|\, dh \\ &\leq D \sum_{\beta \leq \gamma} \left(\int |\, \sigma^{mk} X^\beta \psi(h)\,|^r \, dh \right)^{1/r} \\ &= D \sum_{\beta \leq \gamma} \|\, \psi \,\|^{(r)}_{mk,\beta}, \end{aligned} \tag{6.10}$$

where the $p_\beta((Ad_{h^{-1}})_{ij})$ are polynomials in the matrix entries of $Ad_{h^{-1}}$. So we have $\mathcal{S}^\sigma_r(G) \subseteq \mathcal{S}^\sigma_\infty(G)$.

To see that the inclusion maps are continuous, we use the closed graph theorem just as in Remark 1.2.2 of [Jo]. Let $\psi_n \longrightarrow 0$ in $\mathcal{S}^\sigma_r(G)$, and assume that $\psi_n \longrightarrow \psi$ in $\mathcal{S}^\sigma_\infty(G)$. Assume for a contradiction that $\psi \neq 0$. Then for sufficiently large n, each ψ_n must have absolute value bigger than or equal to some $\delta > 0$ in a fixed neighborhood U in G. But this contradicts $\psi_n \longrightarrow 0$ in $\mathcal{S}^\sigma_r(G)$. So the graph of the inclusion map $\mathcal{S}^\sigma_2(G) \hookrightarrow \mathcal{S}^\sigma_\infty(G)$ must be closed, and thus the inclusion map is continuous. This proves the first statement of the theorem.

For the second statement of the theorem, we use condition (6.9) to show that $\mathcal{S}^\sigma_\infty(G) \subseteq \mathcal{S}^\sigma_r(G)$ if $r < \infty$, with continuous inclusion maps. We let $\psi \in \mathcal{S}^\sigma_\infty(G)$. Then

$$\begin{aligned} \|\, \sigma^m X^\gamma \psi \,\|_r &= \|\, \frac{1}{\sigma^p} \sigma^{m+p} X^\gamma \psi \,\|_r \\ &\leq \|\, \frac{1}{\sigma^p} \,\|_r \|\, \sigma^{m+p} X^\gamma \psi \,\|_\infty \\ &\leq C^{1/r} \|\, \sigma^{m+p} X^\gamma \psi \,\| \end{aligned} \tag{6.11}$$

where C is the value of the integral in (6.9). So the seminorms on $\mathcal{S}_r^\sigma(G)$ are dominated by the seminorms on $\mathcal{S}_\infty^\sigma(G)$. Thus we have the desired continuous inclusions. From the first part of the theorem, it follows that $\mathcal{S}_r^\sigma(G) = \mathcal{S}_1^\sigma(G)$ for any $r \in [1, \infty]$, all with equivalent topologies. □

Corollary 6.12. *If σ is any sub-polynomial scale on a Lie Group G and the identity component G_0 of G is non-trivial, then the group algebra $\mathcal{S}_1^\sigma(G)$ has no (left or right) bounded approximate unit.*

Proof. By Theorem 6.8, any bounded approximate unit must be bounded in the sup norm. But with convolution multiplication, any approximate unit e_n must become unbounded near the identity as $n \to \infty$. □

We show that for many of our examples of gauges and weights, the integrability condition (6.9) is satisfied. Part (2) of the following proposition is proved in [Ji] for discrete groups, and in Proposition 1.5.1 above.

Proposition 6.13. *In each of the following cases, the reciprocal of the scale σ on G is in some L^p-space.*

(1) *The group G is compactly generated and σ is the exponentiated word weight on G.*
(2) *The group G is compactly generated, of polynomial growth, and $\sigma(g) = 1+\tau(g)$, where τ is the word gauge. (Moreover, a gauge which satisfies the integrability condition exists on a compactly generated group G iff G has polynomial growth.)*
(3) *The group G is $GL(n, \mathbb{R})$ and $\sigma(g) = \max(\| g \|, \| g^{-1} \|)$.*
(4) *The group G is a closed subgroup of a Lie group H, and σ is the restriction to G of an H-translationally equivalent scale (also denoted by σ) on H, such that both σ and σ_- bound Ad on H, and $1/\sigma \in L^p(H)$. (For example, σ could be any weight or gauge on H which bounds Ad and satisfies $1/\sigma \in L^p(H)$.)*

Proof. Part (2) is part of Proposition 1.5.1. We prove part (1). Let U be an open rel. comp. generating set for a compactly generated group G. Then $|U^n|$ grows at most exponentially. We see this as follows (compare [Je, p.114- 115]). Since U^2 is rel. comp., and U is open, there is some $l \geq 1$ and finitely many $u_1, \ldots u_k \in U^l$ such that $U^2 \subseteq \cup_{i=1}^k u_i U$. It follows that

Hence the Haar measure of U^n is bounded by k^n times the Haar measure of U.

So let $p \in \mathbb{N}$ be such that the Haar measure of U^n is bounded by $e^{n(p-1)}$. Let τ be the word gauge, and $\omega = e^{\tau}$ be the exponentiated word weight. We have

$$\begin{aligned}\int_G \frac{1}{\omega^p} dg = \int_G e^{-p\tau(g)} dg &\le \sum_{n=0}^{\infty} \int_{U^n} e^{-pn} dg \\ &= \sum_{n=0}^{\infty} |U^n|\, e^{-pn} \le \sum_{n=0}^{\infty} e^{-n} < \infty.\end{aligned}$$

This proves (1).

For (3), use (1) and Example 1.6.10. For (4), assume that σ is defined on a Lie group H which contains G as a closed subgroup, and assume that σ and σ_- bound Ad on H and $1/\sigma \in L^p(H)$. Then σ bounds Ad on G (see Example 1.3.15) so σ bounds the modular function Δ_G (see (4.13) and preceding remarks). Also, σ_- bounds the modular function Δ_H. Let $C > 0$ and $d \in \mathbb{N}$ be such that

$$\Delta_G(g) \le C\sigma^d(g) \qquad g \in G,$$

and

$$\Delta_H(h^{-1}) \le C\sigma^d(h) \qquad h \in H.$$

Since $1/\sigma \in L^p(H)$, by [Bou, chap VII, §2, thm 2] we have

$$\int_G \frac{1}{\sigma^p(hg)} \Xi(g) dg < \infty \tag{6.14}$$

for almost every $h \in H$, where $\Xi(g) = \Delta_H(g)/\Delta_G(g)$. Fix some $h \in H$ such that (6.14) is true. Then since $\Xi(g) \ge 1/(C^2\sigma^{2d}(g))$, we have

$$\begin{aligned}\infty &> \int_G \frac{1}{\sigma^p(hg)} \Xi(g) dg \\ &\ge C_h \int_G \frac{1}{\sigma^{pk}(g)} \Xi(g) dg \qquad \text{trans eq of } \sigma \\ &\ge D \int_G \frac{1}{\sigma^{pk+2d}(g)} dg\end{aligned}$$

for some positive constant D. Hence $1/\sigma \in L^{pk+2d}(G)$. This proves Proposition 6.13. $\square$

Remark 6.15. Since every nilpotent Lie group has polynomial growth [Pa, Cor 6.18], it follows from part (2) of Proposition 6.13 that there exists a gauge on any closed subgroup of a connected nilpotent Lie group which satisfies the integrability condition (6.9)(or (1.5.2)). By

parts (3) and (4) of Proposition 6.13, we see that the weight θ on $GL(n,\mathbb{R})$ defined in Example 1.6.10 satisfies (6.9), and also that the restriction of this weight to $SL(n,\mathbb{R})$ (see Example 1.6.4) or to any closed subgroup of $GL(n,\mathbb{R})$ satisfies (6.9). It also follows from Proposition 6.13, or by simple calculations, that condition (6.9) is satisfied in Examples 1.3.14, 1.6.1, 5.20, and 5.23. Thus it makes no difference in these examples which space $\mathcal{S}_r^\sigma(G)$ of Schwartz functions we use.

Remark 6.16. We compare our algebras with those in [Jo]. (The algebras in [Ji] are special cases of these.) If G is amenable and discrete and σ is a gauge on G, then an easy argument shows that $\mathcal{S}_2^\sigma(G) \subseteq C_r^*(G)$ if and only if $\mathcal{S}_2^\sigma(G) \subseteq l^1(G)$ (see proof of Corollary 3.1.8 of [Jo]). Also, if $\mathcal{S}_2^\sigma(G) \subseteq l^1(G)$, then $\mathcal{S}_2^\sigma(G) \subseteq \mathcal{S}_1^\sigma(G)$ since $\sigma^m\psi(g) \in l^1(G)$ for all $\psi \in \mathcal{S}_2^\sigma(G)$. So if G is amenable, $\mathcal{S}_2^\sigma(G)$ is a subalgebra of $C_r^*(G)$ if and only if $\mathcal{S}_2^\sigma(G) = \mathcal{S}_1^\sigma(G)$.

If G is not amenable, however, it is not clear to me if $\mathcal{S}_2^\sigma(G) \subseteq C_r^*(G)$ implies $\mathcal{S}_2^\sigma(G) \subseteq l^1(G)$. So, as far as I know, there could be cases when $\mathcal{S}_2^\sigma(G)$ is a dense subalgebra of $C_r^*(G)$ and $\mathcal{S}_1^\sigma(G) \neq \mathcal{S}_2^\sigma(G)$. In such a case, Theorem 3.1.7 above does not directly imply the m-convexity of $\mathcal{S}_2^\sigma(G)$. However, the m-convexity of $\mathcal{S}_2^\sigma(G)$ (and in fact all the dense subalgebras in [Jo], including the appendix) is clear from Theorem 3.1.4 above (or [Mi, Prop 4.3]) and the estimates of [Jo, Lemma 1.2.4].

Remark 6.17. Theorem 3.1.7 of [Jo] says that for a discrete group where σ is the word gauge, $\mathcal{S}_2^\sigma(G) \subseteq l^1(G)$ if and only if G is of polynomial growth. But we saw in Remark 6.16 that $\mathcal{S}_2^\sigma(G) \subseteq l^1(G)$ if and only if $\mathcal{S}_2^\sigma(G) \subseteq \mathcal{S}_1^\sigma(G)$, so $\mathcal{S}_1^\sigma(G) = \mathcal{S}_2^\sigma(G)$ if and only if G has polynomial growth. If we drop the hypothesis that σ be a gauge, we can have $\mathcal{S}_1^\sigma(G) = \mathcal{S}_2^\sigma(G)$ even when G does not have polynomial growth. For an example, let G be the free group on two generators, and $\sigma(g) = e^{|g|}$ where $|g|$ denotes the word length of g. Apply Proposition 6.13(1) and Theorem 6.8.

Remark 6.18. We compare our group algebras with Rader's algebra $\mathcal{C}(G)$ [War, vol. II, Prop 8.3.7.14]. Here G is a reductive Lie group - see [War, vol. II, §8.3.7]. A function f is in $\mathcal{C}(G)$ if and only if

$$
{}_{D_1}|f|_{r,D_2} = \sup_{x\in G}(1+\sigma(x))^r \Xi^{-1}(x)|D_1 f D_2(x)| < \infty, \tag{6.19}
$$

where D_1 is a differential operator acting on f on the left, and D_2 is a differential operator acting on f on the right. The Riemannian symmetric function σ is a gauge (in fact equivalent

to the word gauge by Example 1.6.4) on G and Ξ is the zonal spherical function on G. The space $\mathcal{C}(G)$ is then topologized by the seminorms (6.19).

To contrast this with the situation in this paper, we consider the case $G = SL(2,\mathbb{R})$ (see Example 1.6.4). Let $sl(2,\mathbb{R})$ be the Lie algebra of $SL(2,\mathbb{R})$[War]. Let $\| \quad \|_{sl}$ denote the norm on $sl(2,\mathbb{R})$ used in [HW, §2]. (The group $SL(2,\mathbb{R})$ is a special case of the groups considered there.) Let

$$g = \begin{pmatrix} e^a & 0 \\ 0 & e^{-a} \end{pmatrix}.$$

Then by [HW, (2.2a)], we have

$$\sigma(g) = \sigma\left(exp \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \right) = \left\| \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \right\|_{sl} \propto |a|.$$

In (1.6.9), we saw that

$$Ad_g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2a} & 0 \\ 0 & 0 & e^{-2a} \end{pmatrix}.$$

So σ does not bound Ad. So in order to get a *-algebra, we would use some scale other than σ to form our Schwartz algebra. As we saw in Example 1.6.4, the weight $\theta(x) = \max(\| x \|, \| x^{-1} \|)$ defined on $SL(2,\mathbb{R})$ does bound Ad. (In fact, we saw that it is equivalent to the exponentiated word weight.) Here $\| \quad \|$ denotes the norm on a matrix of $SL(2,\mathbb{R})$. Our weight θ grows like the sum of the absolute values of the matrix entries, and thus exponentially faster than σ. Since our weight bounds Ad, we would not gain anything by having differential operators act on both sides as in (6.19). Using Theorem 6.8, our group algebra is the set

$$\text{(6.20)} \qquad \mathcal{S}^\theta_\infty(G) = \{\, f \mid \sup_{x \in G} \theta^r(x) |(Df)(x)| < \infty, \quad r \in \mathbb{N}, \quad D \text{ a diff op } \}.$$

Another difference between the two sets of functions (6.20) and $\mathcal{C}(G)$ is that the seminorms (6.19) for $\mathcal{C}(G)$ make use of the zonal spherical function Ξ, whereas the definition of our space $\mathcal{S}^\theta_\infty(G)$ does not allow that.

As we noted in Remark 6.15, the weight θ satisfies the integrability condition (6.9). To have this condition make sense for σ, we must consider $1+\sigma$, since $\sigma(I) = 0$ (see (1.5.2)). The function $1+\sigma(x)$ does not satisfy (6.9), since otherwise $SL(2,\mathbb{R})$ would have polynomial growth by Proposition 6.13, part 2 (or 1.5.1). And $SL(2,\mathbb{R})$ does not have polynomial growth [Je, Prop 1.3, Thm 1.4].

We note that Rader's algebra is m-convex. By the estimate just preceding Prop 8.3.7.14 of [War, vol II], we have

$$ {}_{D_1}|f * g|_{s,D_2} \leq M\, {}_{D_1}|f|_r |g|_{s,D_2} \tag{6.21} $$

for some constants M and $r \in \mathbb{N}$. Hence the seminorms

$$ \| f \|_l = \max_{|D_1|,|D_2| \leq l} {}_{D_1}|f|_{l,D_2} \tag{6.22} $$

satisfy the condition of Theorem 3.1.4 above and $\mathcal{C}(G)$ is m-convex. (In fact, the m-convexity follows from (6.22) and [Mi, Prop 4.3].)

We show that our group algebra $\mathcal{S}^\theta(G)$ for $SL(2,\mathbb{R})$ is in fact the zero Schwartz space defined in [Bar, §19]. By [Bar, (19.1)], the zero Schwartz space $\mathcal{C}^0(G)$ is topologized by the seminorms

$$ \rho^0_{D_1,D_2,l}(f) = \sup_{x \in G} |e^{(1+l)\sigma(x)} D_1 f D_2(x)|, \tag{6.23} $$

where D_1 and D_2 are differential operators on the right and left respectively, and $l \in \mathbb{N}$. Recall from Example 1.6.4 that $\theta \sim e^\sigma$. Thus our Schwartz space $\mathcal{S}^\theta(G)$ above is topologized by the seminorms

$$ \| f \|_{l,\gamma} = \sup_{x \in G} |e^{(1+l)\sigma(x)} X^\gamma f(x)|, $$

where $l \in \mathbb{N}$. These two topologies are clearly the same, except that (6.23) uses differential operators both on the left and the right. But since e^σ bounds Ad, this makes no difference. Thus $\mathcal{C}^0(G) = \mathcal{S}^\theta(G)$.

We show that if the integrability condition (6.9) holds, then $\mathcal{S}^\sigma(G)$ has the property of being a nuclear Fréchet space[Tr]. If (6.9) fails to hold, then $\mathcal{S}^\sigma_1(G)$ may fail to be nuclear. For example, $l^1(\mathbb{Z})$ is an infinite dimensional Banach space and hence not nuclear [Tr, Cor 50.2].

Theorem 6.24. *Let σ be a translationally equivalent scale on a Lie group G which satisfies the integrability condition (6.9). Then $\mathcal{S}^\sigma_1(G) = \mathcal{S}^\sigma_r(G)$ for all $r \in [1,\infty]$ and all of these spaces are nuclear Fréchet spaces.*

Proof. The first statement is Theorem 6.8. For the second, we imitate the proof of Theorem 6.2.5 of [Pi], which gives the nuclearity for the standard set of Schwartz functions on $\mathbb{R}$

We first recall some general elementary facts and definitions on nuclearity from [Pi]. Let E be any locally convex space. If U is a neighborhood of zero in E (following [Pi], we do not assume neighborhoods are open), then the *polar* U^0 of U denotes the weakly compact subset

$$\{\varphi \in E' | \quad |\varphi(u)| \leq 1, \quad u \in U\} \tag{6.25}$$

of the topological dual E' of E. (The space E' consists of all linear functionals on E which are continuous for the locally convex topology on E.) The seminorm p_U associated with U is defined by

$$p_U(x) = \inf\{\rho > 0 | x \in \rho U\} \tag{6.26}$$

Lemma 6.27 [Pi, Prop 4.1.5]**.** *The locally convex space E is nuclear if and only if some fundamental system $\{U_\alpha\}$ of zero neighborhoods for E has the property that for all $U \in \{U_\alpha\}$ there is some $V \in \{U_\alpha\}$ and a positive Radon measure μ defined on the polar V^0 such that*

$$p_U(x) \leq \int_{V^0} |\varphi'(x)| \, d\mu(\varphi') \tag{6.28}$$

for all $x \in E$.

We use the notation $\mathcal{S}(G)$ for $\mathcal{S}_1^\sigma(G) = \mathcal{S}_\infty^\sigma(G)$. Define seminorms $\| \quad \|_l^1$ and $\| \quad \|_l^\infty$ on $\mathcal{S}(G)$ by

$$\| \psi \|_l^i = \sum_{|\beta| \leq l} \| \sigma^l X^\beta \psi \|_i, \tag{6.29}$$

where $i = 1, \infty$. These give equivalent topologies by assumption (6.9) and Theorem 6.8. We show that the condition for nuclearity given in Lemma 6.27 holds for the fundamental system of L^1 neighborhoods of zero $\{\psi \in \mathcal{S}(G) | \| \psi \|_l^1 \leq 1\}$. Let U_l be the zero neighborhood

$$U_l = \{\psi \in \mathcal{S}(G) | \| \psi \|_l^1 \leq 1\}.$$

(Then $p_{U_l}(\psi) = \| \psi \|_l^1$.) We let V_{l+p} be any L^1 neighborhood of zero contained in the sup norm neighborhood

$$\{\psi \in \mathcal{S}(G) | \| \psi \|_{l+p}^\infty \leq 1\}, \tag{6.30}$$

where p is from condition (6.9).

For each $|\beta| \leq l$ and $g \in G$, the formula

$$\epsilon_g^{[\beta]}(\psi) = \sigma^{l+p}(g)(X^\beta \psi)(g) \tag{6.31}$$

defines a function from $\mathcal{S}(G)$ to $\mathbb{C}$, continuous in ψ. Note that $\epsilon_g^{[\beta]}$ lies in the polar set V_{l+p}^0 of V_{l+p}, since it is in the polar set of (6.30).

Define a positive Radon measure μ on V_{l+p}^0 by the equation

$$\int_{V_{l+p}^0} F(\varphi')d\mu(\varphi') = \sum_{|\beta|\leq l} \int_G \frac{1}{\sigma^p(g)} F(\epsilon_g^{[\beta]})dg \tag{6.32}$$

for all continuous functions F in $C(V_{l+p}^0)$. (Note that condition (6.9) implies that this measure μ is well defined and continuous.) Finally, we have

$$\begin{aligned}
p_{U_l}(\psi) = \| \psi \|_l^1 &= \sum_{|\beta|\leq l} \| \sigma^l X^\beta \psi \|_1 && \text{def (6.28)} \\
&= \sum_{|\beta|\leq l} \| \frac{1}{\sigma^p} \sigma^{l+p} X^\beta \psi \|_1 \\
&= \sum_{|\beta|\leq l} \int_G \frac{1}{\sigma^p(g)} |\epsilon_g^{[\beta]}(\psi)| dg && \text{def of } \epsilon_g^{[\beta]} \\
&= \int_{V_{l+p}^0} |\varphi'(\psi)| d\mu(\varphi') && \text{def of } \mu \text{ (6.32)}
\end{aligned}$$

Thus by Lemma 6.27, the Fréchet space $\mathcal{S}(G)$ is nuclear. This proves Theorem 6.24. □

Question 6.33. Is $\mathcal{S}^\sigma(G)$ nuclear if and only if σ satisfies the integrability condition (6.9)? (See [Jo, Thm 3.1.7] for the discrete case.)

We prove some consequences of our nuclearity result.

Proposition 6.34. *Let E be any Fréchet space. If $\mathcal{S}_1^\sigma(G)$ is nuclear, then $\mathcal{S}_1^\sigma(G, E)$ is isomorphic as a Fréchet space to the projective completion $\mathcal{S}_1^\sigma(G)\widehat{\otimes}_\pi E$.*

Proof. Let $\mathcal{S}(G)$ denote $\mathcal{S}_1^\sigma(G)$, and let $\mathcal{S}(G, E)$ denote $\mathcal{S}^\sigma(G, E)$. Recall that $\mathcal{S}(G, E)$ is the set of C^∞-vectors for the action (2.1.4) of G on $L_1^\sigma(G, E)$, and $\mathcal{S}(G)$ is the set of C^∞-vectors for the the same action of G on $L_1^\sigma(G)$. From this and by Theorem A.8 of the appendix, and the fact that we have a G-equivariant isomorphism $L_1^\sigma(G, E) \cong L_1^\sigma(G)\widehat{\otimes}_\pi E$[Schw, §5], we always have a continuous surjection

$$\theta: \mathcal{S}(G)\widehat{\otimes}_\pi E \to \mathcal{S}(G, E)$$

We show that if $\mathcal{S}(G)$ is nuclear, then this map is injective. Let $\widehat{\otimes}_\epsilon$ denote the completed ϵ tensor product [Tr]. By nuclearity, we have $\mathcal{S}(G)\widehat{\otimes}_\pi E \cong \mathcal{S}(G)\widehat{\otimes}_\epsilon E$. Assume $\theta(\varphi) = 0$ for some $\varphi \in \mathcal{S}(G)\widehat{\otimes}_\epsilon E$. Let e', x' be continuous linear functionals on E and $\mathcal{S}(G)$ respectively. We show that $(x' \otimes e')(\varphi) = 0$. Let $\tilde{e}' \colon \mathcal{S}(G, E) \to \mathcal{S}(G)$ denote the continuous linear map given by $\tilde{e}'(\psi)(g) = e'(\psi(g))$ for $\psi \in \mathcal{S}(G, E)$. Then $0 = \tilde{e}'(\theta(\varphi)) = (id \otimes e')(\varphi)$. But $(x' \otimes e')(\varphi) = x'((id \otimes e')(\varphi))$. Hence $(x' \otimes e')(\varphi) = 0$ for all pairs $(x', e') \in \mathcal{S}(G)' \times E'$. By the definition of the completed epsilon tensor product $\mathcal{S}(G)\widehat{\otimes}_\epsilon E$ [Tr, Def 43.5, 43.1], this implies that $\varphi = 0$. Hence θ is injective and we have proved Proposition 6.34. $\square$

Theorem 6.35. *Let (M, σ, H) be a scaled (G, ω)-space. Assume that $1/\omega$ is in $L^p(G)$ for some p. Let $\mathcal{S}(M)$ denote $\mathcal{S}^\sigma_H(M)$, and let $\mathcal{S}(G \times M)$ denote $\mathcal{S}^{\omega\times\sigma}_{G\times H}(G \times M)$. We then have*

$$\mathcal{S}(G)\widehat{\otimes}_\pi \mathcal{S}(M) \cong \mathcal{S}(G \times M). \tag{6.36}$$

By Proposition 6.34 above, we have $G \rtimes \mathcal{S}(M) \cong \mathcal{S}(G \times M)$, so that we may view $G \rtimes \mathcal{S}(M)$ as a space of $G \times H$-differentiable $\omega \times \sigma$-rapidly vanishing functions.

Proof. We show this by an argument analogous to [Tr, Thm 51.6], using the fact that $\mathcal{S}(G)$ is nuclear. A simple calculation shows that the map

$$\mathcal{S}(G) \times \mathcal{S}(M) \longrightarrow \mathcal{S}(G \times M)$$

$$(\psi, \phi) \mapsto \psi(g)\phi(m)$$

is a continuous bilinear map. Thus we have an inclusion

$$\mathcal{S}(G) \otimes \mathcal{S}(M) \hookrightarrow \mathcal{S}(G \times M),$$

continuous if we place the projective topology on the uncompleted tensor product. We will be done if we show that $\mathcal{S}(G \times M)$ induces a topology on $\mathcal{S}(G) \otimes \mathcal{S}(M)$ which is stronger than (and hence equivalent to) the projective topology, and that $\mathcal{S}(G) \otimes \mathcal{S}(M)$ is dense in $\mathcal{S}(G \times M)$.

Dense. For the denseness, we use the fact that the algebraic tensor product of continuous functions $C_c(G) \otimes C_c(M)$ is dense in $C_c(G \times M)$ in the inductive limit topology. It follows that $C^\omega(G) \otimes C^\sigma(M)$ is dense in $C^{\omega\times\sigma}(G \times M)$, so that the canonical map $C^\omega(G)\widehat{\otimes}_\pi C^\sigma(M) \longrightarrow C^{\omega\times\sigma}(G \times M)$ has dense image. Using Theorem A.8 and the definitions of our spaces, we see that $\mathcal{S}(G)\widehat{\otimes}_\pi \mathcal{S}(M) \longrightarrow \mathcal{S}(G \times M)$ has dense image. Thus the algebraic tensor product $\mathcal{S}(G) \otimes \mathcal{S}(M)$ is dense in $\mathcal{S}(G \times M)$.

Topologies equivalent. Since $\mathcal{S}(G)$ is nuclear, the projective topology on $\mathcal{S}(G) \otimes \mathcal{S}(M)$ agrees with the ϵ-topology. So it suffices to show that if a sequence Ξ_n in $\mathcal{S}(G) \otimes \mathcal{S}(M)$ converges to zero in $\mathcal{S}(G \times M)$, then it converges to zero in the ϵ-topology. For this it suffices to show that if $A \subseteq \mathcal{S}(G)'$ and $B \subseteq \mathcal{S}(M)'$ are equicontinuous, then $\Xi_n \longrightarrow 0$ uniformly on $A \otimes B \subseteq \mathcal{S}(G \times M)'$ [Tr]. (Here, $A \otimes B$ denotes the set $\{a \otimes b | a \in A, \quad b \in B\}$.) This is true by the following lemma.

Lemma 6.37. *If $A \subseteq \mathcal{S}(G)'$ and $B \subseteq \mathcal{S}(M)'$ are equicontinuous, then $A \otimes B \subseteq \mathcal{S}(G \times M)'$ is an equicontinuous subset of $\mathcal{S}(G \times M)'$.*

Proof. Note that $\mathcal{S}(G \times M)$ is topologized by the seminorms

$$\| \omega^p \sigma^q X^\gamma \tilde{X}^\beta \Xi \|_\infty = \sup_{x \in G, m \in M} \omega^p(x) \sigma^q(m) |X^\gamma \tilde{X}^\beta \Xi(g, m)| \tag{6.38}$$

where X^γ acts on the first argument of Ξ, and $\tilde{X}^\beta$ on the second. By equicontinuity, let p, q and C, D be such that

$$|S(\phi)| \le C \max_{|\gamma| \le p} \| \omega^p X^\gamma \phi \|_\infty, \qquad \phi \in \mathcal{S}(G), \tag{6.39}$$

and

$$|T(\psi)| \le D \max_{|\beta| \le q} \| \sigma^q \tilde{X}^\beta \psi \|_\infty, \qquad \psi \in \mathcal{S}(M), \tag{6.40}$$

for all $S \in A$ and $T \in B$. We follow the argument of [Hor, Lemma 3, P. 371] to show that

$$|(S \otimes T)(\Xi)| \le CD \max_{|\gamma| \le p, |\beta| \le q} \| \omega^p \sigma^q X^\gamma \tilde{X}^\beta \Xi \|_\infty, \qquad \Xi \in \mathcal{S}(G \times M), \tag{6.41}$$

from which the lemma follows.

We introduce the integral notation

$$S(\phi) = \int_G S(x) \phi(x) dx$$

$$T(\psi) = \int_M T(m) \psi(m) dm.$$

Then if $\Xi \in \mathcal{S}(G \times M)$,

$$| \int S(x) \tilde{X}^\beta \Xi(x, m) dx | \le C \max \sup \omega^p(x) |X^\gamma \tilde{X}^\beta \Xi(x, m)| \tag{6.42}$$

by (6.39). So we have

(6.43)
$$
\begin{aligned}
|(S\otimes T)\Xi| &= |\int_M T(m)\Big(\int_G S(x)\Xi(x,m)dx\Big)dm| \\
&\le D \max_{|\beta|\le q}\sup_{m\in M}\sigma^q(m)|\tilde{X}^\beta \int_G S(x)\Xi(x,m)dx| \qquad (6.40)\\
&= D \max_{|\beta|\le q}\sup_{m\in M} |\int_G \sigma^q(m) S(x)\tilde{X}^\beta \Xi(x,m)dx| \\
&\le CD \max_{|\gamma|\le p, |\beta|\le q} \| \omega^p\sigma^q X^\gamma \tilde{X}^\beta \Xi \|_\infty
\end{aligned}
$$

by (6.42). This proves the lemma, and the isomorphism (6.36). □

Appendix. Sets of C^∞-vectors.

We show that the set of C^∞-vectors E^∞ for the action of a Lie group G on a Fréchet space E is a Fréchet space for the natural topology, and that G leaves E^∞ invariant and acts differentiably on E^∞ by continuous automorphisms. If E is an m-convex Fréchet *-algebra, we show that E^∞ is also. These facts are well known [DM], but the literature seems to prove them only for Banach or Hilbert spaces [Go], [Po], [Ta]. We conclude the appendix with a technical lemma which is useful in Proposition 2.2.8, Proposition 6.34 and Theorem 6.35.

Let α denote the action of G on E. We assume that α gives a strongly continuous action of G on E by continuous automorphisms. We define the set of C^∞-vectors E^∞ for the action of G on E, to be the set of $e \in E$ such that all the derivatives $X^\gamma e$ (see (1.2.1) with $\beta = \alpha$) exist for the topology of E. Here q is the dimension of the Lie algebra $\mathfrak{G}$ of G and $\gamma \in \mathbb{N}^q$. Let $\| \ \|_m$ be a family of seminorms topologizing E. Then we topologize E^∞ by the seminorms

(A.1)
$$\| e \|_{l,m} = \max_{|\gamma|\le l} \| X^\gamma e \|_m$$

Theorem A.2. *For the topology given by the seminorms $\| \ \|_{l,m}$, the locally convex space E^∞ is complete. Thus E^∞ is a Fréchet space. In the topology of E, the set E^∞ is dense. The action α of G on E leaves E^∞ invariant, and each α_g restricts to a continuous automorphism of E^∞. For fixed $e \in E^\infty$, the map $g \mapsto \alpha_g(e)$ is infinitely differentiable from G to E^∞ (or equivalently, all the derivatives of elements of E^∞ converge in the topology of E^∞). If E is an (m-convex) Fréchet [*]-algebra, and G acts by [*]-automorphisms on E, then E^∞ is an (m-convex) Fréchet [*]-algebra.*

Proof. Let e_m be a Cauchy sequence in E^∞, and for each $\gamma \in \mathbb{N}^q$ let $e^\gamma \in E$ be such that

(A.3)
$$X^\gamma e_m \to e^\gamma \quad \text{in } E$$

If $\gamma = (0, \ldots, 0)$, let e denote e^{γ}. We show $e_m \longrightarrow e$ in E^{∞}. Let $\gamma_i = (0, \ldots 1, 0, \ldots)$, with a 1 in the ith spot. Then

$$
\begin{aligned}
\int_0^t \alpha_{exptX_i}(e^{\gamma_i})dt &= \lim_m \int_0^t \alpha_{exptX_i}(X^{\gamma_i} e_m)dt && \text{(A.3) and unif conv} \\
&= \lim_m \alpha_{exptX_i}(e_m) - e_m && \text{Fund thm calc} \\
&= \alpha_{exptX_i}(e) - e
\end{aligned}
\tag{A.4}
$$

so the derivative $X^{\gamma_i} e$ exists in the topology of E and equals e^{γ_i}. Repeated application of this argument shows that $X^{\gamma} e$ exists in E for all $\gamma \in \mathbb{N}^q$, and equals e^{γ}. By (A.3), this implies $e_m \longrightarrow e$ in the topology of E^{∞}. Thus E^{∞} is complete, and a Fréchet space. We remark that the completeness is shown in [Go, Cor 1.1] for E a Hilbert space.

The argument of [Ta, p. 11] shows that if G acts strongly continuously on E, then E^{∞} is dense in E. The simple argument indicated in [Po, §1] shows that E^{∞} is G-invariant. Also, the proof of [Po, Prop 1.2] works for Fréchet spaces, so $g \mapsto \alpha_g(e)$ is differentiable.

Now assume that E is a Fréchet algebra, and that α_g is an algebra automorphism of E for each $g \in G$. Let $\gamma \in \mathbb{N}^q$ and $e, f \in E$. By the product rule,

$$
X^{\gamma}(ef) = \sum_{\beta, \tilde{\beta}} c_{\beta, \tilde{\beta}} (X^{\beta} e)(X^{\tilde{\beta}} f),
\tag{A.5}
$$

where β and $\tilde{\beta}$ have order less than or equal to $|\gamma|$, and $c_{\beta,\tilde{\beta}}$ are appropriate constants. We have

$$
\begin{aligned}
\| ef \|_{l,p} &= \max_{|\gamma| \le l} \| X^{\gamma}(ef) \|_p && \text{for the seminorms (A.1)} \\
&\le \sum_{\beta, \tilde{\beta}} c_{\beta, \tilde{\beta}} \| (X^{\beta} e)(X^{\tilde{\beta}} f) \|_p, && \text{by (A.5)} \\
&\le \sum_{\beta, \tilde{\beta}} \tilde{c}_{\beta, \tilde{\beta}} \| (X^{\beta} e) \|_d \| (X^{\tilde{\beta}} f) \|_d && E \text{ Fréchet alg} \\
&\le C \| e \|_{l,d} \| f \|_{l,d}, && \text{by def of norms (A.1)}
\end{aligned}
\tag{A.6}
$$

for some $C > 0$ depending only on l and d. Thus E^{∞} is a Fréchet algebra. If E is m-convex, then we may take $d = p$ in (A.6), so the m-convexity of E^{∞} clearly follows (without using Theorem 3.1.4). If E is a Fréchet *-algebra and G acts by *-automorphisms on E, then $X^{\gamma} e^* = (X^{\gamma} e)^*$ for all $e \in \mathbb{N}$. Hence the * operation will be continuous on the seminorms (A.1) and E^{∞} is a Fréchet *-algebra. This proves Theorem A.2. $\square$

Let F be any Fréchet space. Then G has a natural continuous action on the completed projective tensor product $E \hat{\otimes}_{\pi} F$ given by

$$
\alpha_g(e \otimes f) = \alpha_g(e) \otimes f
\tag{A.7}
$$

on elementary tensors. It is easily checked using the absolutely convergent series expression for elements of $E\widehat{\otimes}_\pi F$ in [Tr, Thm 45.1] that this gives a strongly continuous action of G. The surjectivity in the following theorem uses the theorem of Dixmier and Malliavin [DM, Thm 3.3].

Theorem A.8. *Let C_α^∞ denote the set of C^∞-vectors for the action of α on $E\hat{\otimes}_\pi F$. Then there is a continuous surjection $E^\infty\hat{\otimes}_\pi F \longrightarrow C_\alpha^\infty$ of Fréchet spaces.*

Proof. Since we shall always be dealing with the projective tensor product in this theorem, we omit the subscript π from $\hat{\otimes}_\pi$. We have a natural continuous map $\pi\colon E^\infty\hat{\otimes}F \longrightarrow E\hat{\otimes}F$. We show that the image of π is contained in C_α^∞. If $x \in E^\infty\hat{\otimes}F$, we may write x as an absolutely convergent series $\sum \lambda_n e_n \otimes f_n$, where $e_n \longrightarrow 0$ in E^∞, $f_n \longrightarrow 0$ in F, and $\sum|\lambda_n| < 1$ [Tr, Thm 45.1]. Let X be in the Lie algebra of G. If $z \in C_\alpha^\infty$, then Xz is the limit

$$Xz = \lim_{t\to 0} \frac{\alpha_{(exptX)}(z) - z}{t} \tag{A.9}$$

We show that if we plug the series expression for x in place of z into (A.9), then the limit converges to an element of $E\hat{\otimes}F$ in the topology of $E\hat{\otimes}F$. This will imply that $x \in C_\alpha^\infty$. Define

$$\psi(t)(e) = \alpha_{(exptX)}(e)$$

We abbreviate $\psi(t)(e_n)$ by $\psi_n(t)$. Since $e_n \longrightarrow 0$ in E^∞, $\psi_n'(0)$ converges to zero in E^∞. Define

$$x' \equiv \sum_{n=0}^{\infty} \lambda_n \psi_n'(0) \otimes f_n \tag{A.10}$$

The series clearly converges absolutely in $E\hat{\otimes}F$ and so $x' \in E\hat{\otimes}F$. We proceed to show that the limit (A.9) with $z = x$ converges in $E\hat{\otimes}F$ to x'. By elementary calculus

$$\psi_n(t) = \psi_n(0) + t\psi_n'(0) + t^2 \int_0^1 (1-s)\psi_n''(ts)ds. \tag{A.11}$$

Subtracting (A.10) from (A.9) with the series for x plugged in for z, we see that $Xx - x'$ is the limit (if it exists), as t tends to zero, of the infinite series

$$\begin{aligned} &\sum_{n=0}^{\infty} \lambda_n \Big(\frac{\psi_n(t) - \psi_n(0)}{t} - \psi_n'(0) \Big) \otimes f_n = \quad \text{by (A.11)} \\ &\sum_{n=0}^{\infty} \lambda_n \Big(t \int_0^1 (1-s)\psi_n''(ts)ds \Big) \otimes f_n. \end{aligned} \tag{A.12}$$

Let $\| \quad \|_d$, $\| \quad \|_l$ be norms for the topology on E, F respectively and let $\| \quad \|_{d,l}$ be the induced norm on the tensor product. Let $T_t \colon E^\infty \longrightarrow E$ be the continuous linear operator $T_t(e) = \psi''(t)(e) - X^2 e$. Then T_t converges to zero pointwise as $t \longrightarrow 0$. By the uniform boundedness principle for Fréchet spaces[RS, Thm V.7], there is some $k, p \in \mathbb{N}$, $C > 0$ such that

$$\| T_t e \|_d \leq C \max_{|\gamma| \leq p} \| X^\gamma e \|_k \tag{A.13}$$

for $t \in [0,1]$, $e \in E^\infty$. It follows that

$$\| \psi_n''(t) \|_d \leq C \max_{|\gamma| \leq p} \| X^\gamma e_n \|_k + \| X^2 e_n \|_d \, .$$

for all $t \in [0,1]$ and $n \in \mathbb{N}$. Since the terms on the right hand side tend to zero as $n \longrightarrow \infty$, there is some $D > 0$ such that

$$\sup_{t \in [0,1], n \in \mathbb{N}} \| \psi_n''(t) \|_d < D. \tag{A.14}$$

The right hand side of (A.12) evaluated at $\| \quad \|_{d,l}$ is then less than or equal to

$$|t| \sum_{n=0}^{\infty} \lambda_n D \| f_n \|_l \tag{A.15}$$

if $|t| < 1$. This clearly tends to zero as t tends to zero. Thus the limit Xx converges to x' in $E \hat{\otimes} F$. It follows that π maps $E^\infty \hat{\otimes} F$ continuously into C_α^∞.

We now prove that $C_\alpha^\infty \subset \pi(E^\infty \hat{\otimes} F)$. By [DM, Thm 3.3] any element of C_α^∞ is a finite sum of elements of the form $\alpha_f(y)$, where $f \in C_c^\infty(G)$ and $y \in E \hat{\otimes} F$. By definition, the expression $\alpha_f(y)$ is the integral

$$\int_G f(g) \alpha_g(y) dg. \tag{A.16}$$

If we write y as an absolutely convergent series of elementary tensors (as in [Tr, Thm 45.1]) and take the integral inside the sum, we get a series converging absolutely in $E^\infty \hat{\otimes} F$. Hence the series for $\alpha_f(y)$ converges in $E^\infty \hat{\otimes} F$. Since π maps this series to $\alpha_f(y)$ in $E \hat{\otimes} F$, we have proved that $C_\alpha^\infty \subseteq \pi(E^\infty \hat{\otimes} F)$. This proves Theorem A.8. $\square$

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